If the particles have a gravitational interaction, then $n = -2$, and

$$\langle T \rangle = -\frac{1}{2}\langle U \rangle, \quad n = -2$$

This relation is useful in calculating, for example, the energetics in planetary motion.

PROBLEMS

7-1. A disk rolls without slipping across a horizontal plane. The plane of the disk remains vertical, but it is free to rotate about a vertical axis. What generalized coordinates may be used to describe the motion? Write a differential equation describing the rolling constraint. Is this equation integrable? Justify your answer by a physical argument. Is the constraint holonomic?

7-2. Work out Example 7.6 showing all the steps, in particular those leading to Equations 7.36 and 7.41. Explain why the sign of the acceleration a cannot affect the frequency ω. Give an argument why the signs of a^2 and g^2 in the solution of ω^2 in Equation 7.42 are the same.

7-3. A sphere of radius ρ is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius R. Determine the Lagrangian function, the equation of constraint, and Lagrange's equations of motion. Find the frequency of small oscillations.

7-4. A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; A and α (> 0) are constants. Choose appropriate generalized coordinates, and let the potential energy be zero at the origin. Find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Is the total energy conserved?

7-5. Consider a vertical plane in a constant gravitational field. Let the origin of a coordinate system be located at some point in this plane. A particle of mass m moves in the vertical plane under the influence of gravity and under the influence of an additional force $f = -Ar^{\alpha-1}$ directed toward the origin (r is the distance from the origin; A and α [$\neq 0$ or 1] are constants). Choose appropriate generalized coordinates, and find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Explain.

7-6. A hoop of mass m and radius R rolls without slipping down an inclined plane of mass M, which makes an angle α with the horizontal. Find the Lagrange equations and the integrals of the motion if the plane can slide without friction along a horizontal surface.

7-7. A double pendulum consists of two simple pendula, with one pendulum suspended from the bob of the other. If the two pendula have equal lengths and have bobs of equal mass and if both pendula are confined to move in the same plane, find Lagrange's equations of motion for the system. Do not assume small angles.

7-8. Consider a region of space divided by a plane. The potential energy of a particle in region 1 is U_1 and in region 2 it is U_2. If a particle of mass m and with speed v_1 in region 1 passes from region 1 to region 2 such that its path in region 1 makes an angle θ_1 with the normal to the plane of separation and an angle θ_2 with the normal when in region 2, show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \left(1 + \frac{U_1 - U_2}{T_1}\right)^{1/2}$$

where $T_1 = \frac{1}{2}mv_1^2$. What is the optical analog of this problem?

7-9. A disk of mass M and radius R rolls without slipping down a plane inclined from the horizontal by an angle α. The disk has a short weightless axle of negligible radius. From this axis is suspended a simple pendulum of length $l < R$ and whose bob has a mass m. Consider that the motion of the pendulum takes place in the plane of the disk, and find Lagrange's equations for the system.

7-10. Two blocks, each of mass M, are connected by an extensionless, uniform string of length l. One block is placed on a smooth horizontal surface, and the other block hangs over the side, the string passing over a frictionless pulley. Describe the motion of the system **(a)** when the mass of the string is negligible and **(b)** when the string has a mass m.

7-11. A particle of mass m is constrained to move on a circle of radius R. The circle rotates in space about one point on the circle, which is fixed. The rotation takes place in the plane of the circle and with constant angular speed ω. In the absence of a gravitational force, show that the particle's motion about one end of a diameter passing through the pivot point and the center of the circle is the same as that of a plane pendulum in a uniform gravitational field. Explain why this is a reasonable result.

7-12. A particle of mass m rests on a smooth plane. The plane is raised to an inclination angle θ at a constant rate α ($\theta = 0$ at $t = 0$), causing the particle to move down the plane. Determine the motion of the particle.

7-13. A simple pendulum of length b and bob with mass m is attached to a massless support moving horizontally with constant acceleration a. Determine **(a)** the equations of motion and **(b)** the period for small oscillations.

7-14. A simple pendulum of length b and bob with mass m is attached to a massless support moving vertically upward with constant acceleration a. Determine **(a)** the equations of motion and **(b)** the period for small oscillations.

7-15. A pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k. Find Lagrange's equations of motion.

7-16. The point of support of a simple pendulum of mass m and length b is driven horizontally by $x = a\sin\omega t$. Find the pendulum's equation of motion.

7-17. A particle of mass m can slide freely along a wire AB whose perpendicular distance to the origin O is h (see Figure 7-A, page 282). The line OC rotates about the origin

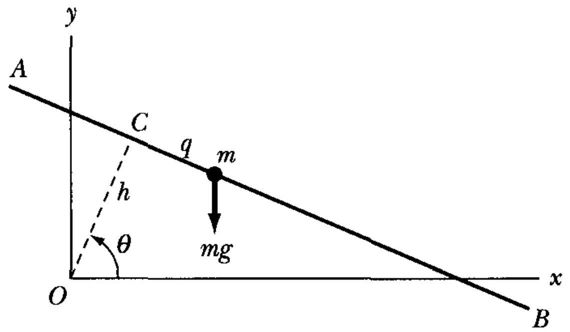

FIGURE 7-A Problem 7-17.

at a constant angular velocity $\dot{\theta} = \omega$. The position of the particle can be described in terms of the angle θ and the distance q to the point C. If the particle is subject to a gravitational force, and if the initial conditions are

$$\theta(0) = 0, \quad q(0) = 0, \quad \dot{q}(0) = 0$$

show that the time dependence of the coordinate q is

$$q(t) = \frac{g}{2\omega^2}(\cosh \omega t - \cos \omega t)$$

Sketch this result. Compute the Hamiltonian for the system, and compare with the total energy. Is the total energy conserved?

7-18. A pendulum is constructed by attaching a mass m to an extensionless string of length l. The upper end of the string is connected to the uppermost point on a vertical disk of radius R ($R < l/\pi$) as in Figure 7-B. Obtain the pendulum's equation of motion, and find the frequency of small oscillations. Find the line about which the angular motion extends equally in either direction (i.e., $\theta_1 = \theta_2$).

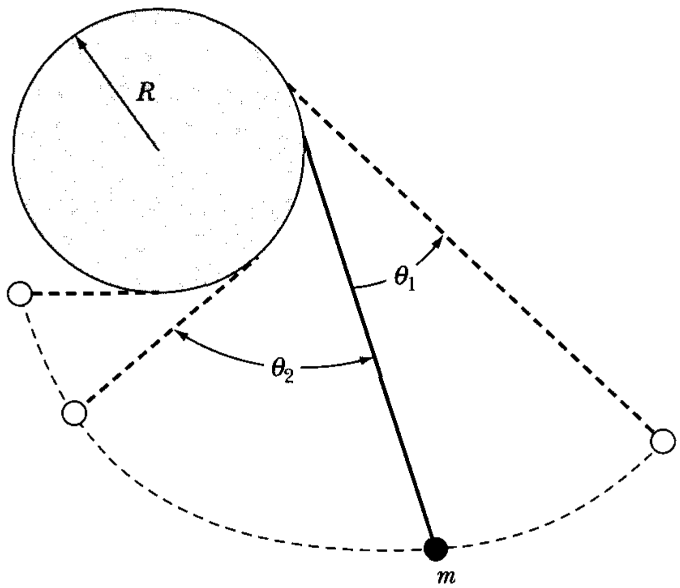

FIGURE 7-B Problem 7-18.

7-19. Two masses m_1 and m_2 ($m_1 \neq m_2$) are connected by a rigid rod of length d and of negligible mass. An extensionless string of length l_1 is attached to m_1 and connected to a fixed point of support P. Similarly, a string of length l_2 ($l_1 \neq l_2$) connects m_2 and P. Obtain the equation describing the motion in the plane of m_1, m_2, and P, and find the frequency of small oscillations around the equilibrium position.

7-20. A circular hoop is suspended in a horizontal plane by three strings, each of length l, which are attached symmetrically to the hoop and are connected to fixed points lying in a plane above the hoop. At equilibrium, each string is vertical. Show that the frequency of small rotational oscillations about the vertical through the center of the hoop is the same as that for a simple pendulum of length l.

7-21. A particle is constrained to move (without friction) on a circular wire rotating with constant angular speed ω about a vertical diameter. Find the equilibrium position of the particle, and calculate the frequency of small oscillations around this position. Find and interpret physically a critical angular velocity $\omega = \omega_c$ that divides the particle's motion into two distinct types. Construct phase diagrams for the two cases $\omega < \omega_c$ and $\omega > \omega_c$.

7-22. A particle of mass m moves in one dimension under the influence of a force

$$F(x, t) = \frac{k}{x^2} e^{-(t/\tau)}$$

where k and τ are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

7-23. Consider a particle of mass m moving freely in a conservative force field whose potential function is U. Find the Hamiltonian function, and show that the canonical equations of motion reduce to Newton's equations. (Use rectangular coordinates.)

7-24. Consider a simple plane pendulum consisting of a mass m attached to a string of length l. After the pendulum is set into motion, the length of the string is shortened at a constant rate

$$\frac{dl}{dt} = -\alpha = \text{constant}$$

The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

7-25. A particle of mass m moves under the influence of gravity along the helix $z = k\theta$, $r =$ constant, where k is a constant and z is vertical. Obtain the Hamiltonian equations of motion.

7-26. Determine the Hamiltonian and Hamilton's equations of motion for **(a)** a simple pendulum and **(b)** a simple Atwood machine (single pulley).

7-27. A massless spring of length b and spring constant k connects two particles of masses m_1 and m_2. The system rests on a smooth table and may oscillate and rotate.

(a) Determine Lagrange's equations of motion.
(b) What are the generalized momenta associated with any cyclic coordinates?
(c) Determine Hamilton's equations of motion.

7-28. A particle of mass m is attracted to a force center with the force of magnitude k/r^2. Use plane polar coordinates and find Hamilton's equations of motion.

7-29. Consider the pendulum described in Problem 7-15. The pendulum's point of support rises vertically with constant acceleration a.
(a) Use the Lagrangian method to find the equations of motion.
(b) Determine the Hamiltonian and Hamilton's equations of motion.
(c) What is the period of small oscillations?

7-30. Consider any two continuous functions of the generalized coordinates and momenta $g(q_k, p_k)$ and $h(q_k, p_k)$. The **Poisson brackets** are defined by

$$[g, h] \equiv \sum_k \left(\frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_k}\right)$$

Verify the following properties of the Poisson brackets:

(a) $\dfrac{dg}{dt} = [g, H] + \dfrac{\partial g}{dt}$ **(b)** $\dot{q}_j = [q_i, H],\ \dot{p}_j = [p_j, H]$

(c) $[p_l, p_j] = 0,\ [q_l, q_j] = 0$ **(d)** $[q_l, p_j] = \delta_{lj}$

where H is the Hamiltonian. If the Poisson bracket of two quantities vanishes, the quantities are said to *commute.* If the Poisson bracket of two quantities equals unity, the quantities are said to be *canonically conjugate.* **(e)** Show that any quantity that does not depend explicitly on the time and that commutes with the Hamiltonian is a constant of the motion of the system. Poisson-bracket formalism is of considerable importance in quantum mechanics.

7-31. A spherical pendulum consists of a bob of mass m attached to a weightless, extensionless rod of length l. The end of the rod opposite the bob pivots freely (in all directions) about some fixed point. Set up the Hamiltonian function in spherical coordinates. (If $p_\phi = 0$, the result is the same as that for the plane pendulum.) Combine the term that depends on p_ϕ with the ordinary potential energy term to define as *effective* potential $V(\theta, p_\phi)$. Sketch V as a function of θ for several values of p_ϕ, including $p_\phi = 0$. Discuss the features of the motion, pointing out the differences between $p_\phi = 0$ and $p_\phi \neq 0$. Discuss the limiting case of the conical pendulum (θ = constant) with reference to the V-θ diagram.

7-32. A particle moves in a spherically symmetric force field with potential energy given by $U(r) = -k/r$. Calculate the Hamiltonian function in spherical coordinates, and obtain the canonical equations of motion. Sketch the path that a representative point for the system would follow on a surface H = constant in phase space. Begin by showing that the motion must lie in a plane so that the phase space is four dimensional (r, θ, p_r, p_θ, but only the first three are nontrivial). Calculate the projection of the phase path on the r-p_r plane, then take into account the variation with θ.

7-33. Determine the Hamiltonian and Hamilton's equations of motion for the double Atwood machine of Example 7.8.

7-34. A particle of mass m slides down a smooth circular wedge of mass M as shown in Figure 7-C. The wedge rests on a smooth horizontal table. Find (**a**) the equation of motion of m and M and (**b**) the reaction of the wedge on m.

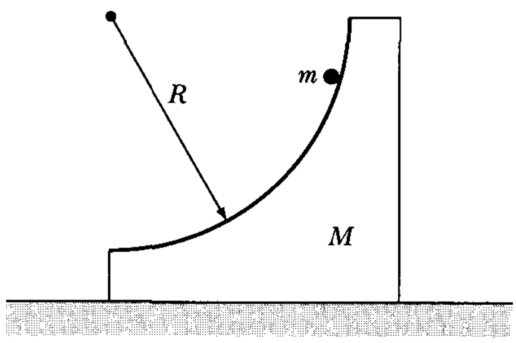

FIGURE 7-C Problem 7-34.

7-35. Four particles are directed upward in a uniform gravitational field with the following initial conditions:

$$\begin{array}{lll} \textbf{(1)} & z(0) = z_0; & p_z(0) = p_0 \\ \textbf{(2)} & z(0) = z_0 + \Delta z_0; & p_z(0) = p_0 \\ \textbf{(3)} & z(0) = z_0; & p_z(0) = p_0 + \Delta p_0 \\ \textbf{(4)} & z(0) = z_0 + \Delta z_0; & p_z(0) = p_0 + \Delta p_0 \end{array} .$$

Show by direct calculation that the representative points corresponding to these particles always define an area in phase space equal to $\Delta z_0 \, \Delta p_0$. Sketch the phase paths, and show for several times $t > 0$ the shape of the region whose area remains constant.

7-36. Discuss the implications of Liouville's theorem on the focusing of beams of charged particles by considering the following simple case. An electron beam of circular cross section (radius R_0) is directed along the z-axis. The density of electrons across the beam is constant, but the momentum components transverse to the beam (p_x and p_y) are distributed uniformly over a circle of radius p_0 in momentum space. If some focusing system reduces the beam radius from R_0 to R_1, find the resulting distribution of the transverse momentum components. What is the physical meaning of this result? (Consider the angular divergence of the beam.)

7-37. Use the method of Lagrange undetermined multipliers to find the tensions in both strings of the double Atwood machine of Example 7.8.

7-38. The potential for an anharmonic oscillator is $U = kx^2/2 + bx^4/4$ where k and b are constants. Find Hamilton's equations of motion.

7-39. An extremely limber rope of uniform mass density, mass m and total length b lies on a table with a length z hanging over the edge of the table. Only gravity acts on the rope. Find Lagrange's equation of motion.

7-40. A double pendulum is attached to a cart of mass $2m$ that moves without friction on a horizontal surface. See Figure 7-D. Each pendulum has length b and mass bob m. Find the equations of motion.

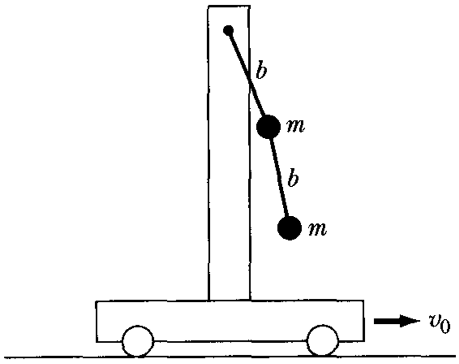

FIGURE 7-D Problem 7-40.

7-41. A pendulum of length b and mass bob m is oscillating at small angles when the length of the pendulum string is shortened at a velocity of α $(db/dt = -\alpha)$. Find Lagrange's equations of motion.