PROBLEMS

6-1. Consider the line connecting $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 1)$. Show explicitly that the function $y(x) = x$ produces a minimum path length by using the varied function $y(\alpha, x) = x + \alpha \sin \pi(1 - x)$. Use the first few terms in the expansion of the resulting elliptic integral to show the equivalent of Equation 6.4.

6-2. Show that the shortest distance between two points on a plane is a straight line.

6-3. Show that the shortest distance between two points in (three-dimensional) space is a straight line.

6-4. Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.

6-5. Consider the surface generated by revolving a line connecting two fixed points (x_1, y_1) and (x_2, y_2) about an axis coplanar with the two points. Find the equation of the line connecting the points such that the surface area generated by the revolution (i.e., the area of the surface of revolution) is a minimum. Obtain the solution by using Equation 6.39.

6-6. Reexamine the problem of the brachistochrone (Example 6.2) and show that the time required for a particle to move (frictionlessly) to the *minimum* point of the cycloid is $\pi\sqrt{a/g}$, *independent* of the starting point.

6-7. Consider light passing from one medium with index of refraction n_1 into another medium with index of refraction n_2 (Figure 6-A). Use Fermat's principle to minimize time, and derive the law of refraction: $n_1 \sin \theta_1 = n_2 \sin \theta_2$.

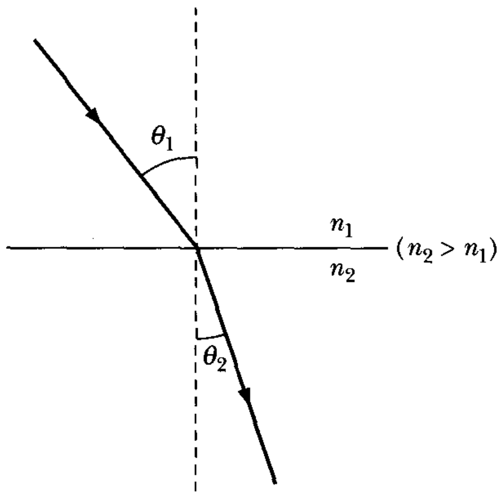

FIGURE 6-A Problem 6-7.

6-8. Find the dimensions of the parallelepiped of maximum volume circumscribed by **(a)** a sphere of radius R; **(b)** an ellipsoid with semiaxes a, b, c.

6-9. Find an expression involving the function $\phi(x_1, x_2, x_3)$ that has a minimum average value of the square of its gradient within a certain volume V of space.

6-10. Find the ratio of the radius R to the height H of a right-circular cylinder of fixed volume V that minimizes the surface area A.

6-11. A disk of radius R rolls without slipping inside the parabola $y = ax^2$. Find the equation of constraint. Express the condition that allows the disk to roll so that it contacts the parabola at one and only one point, independent of its position.

6-12. Repeat Example 6.4, finding the shortest path between any two points on the surface of a sphere, but use the method of the Euler equations with an auxiliary condition imposed.

6-13. Repeat Example 6.6 but do not use the constraint that the $y = 0$ line is the bottom part of the area. Show that the plane curve of a given length, which encloses a maximum area, is a circle.

6-14. Find the shortest path between the (x, y, z) points $(0, -1, 0)$ and $(0, 1, 0)$ on the conical surface $z = 1 - \sqrt{x^2 + y^2}$. What is the length of the path? Note: this is the shortest mountain path around a volcano.

6-15. **(a)** Find the curve $y(x)$ that passes through the endpoints $(0, 0)$ and $(1, 1)$ and minimizes the functional $I[y] = \int_0^1 [(dy/dx)^2 - y^2]\,dx$. **(b)** What is the minimum value of the integral? **(c)** Evaluate $I[y]$ for a straight line $y = x$ between the points $(0, 0)$ and $(1, 1)$.

6-16. **(a)** What curve on the surface $z = x^{3/2}$ joining the points $(x, y, z) = (0, 0, 0)$ and $(1, 1, 1)$ has the shortest arc length? **(b)** Use a computer to produce a plot showing the surface and the shortest curve on a single plot.

6-17. The corners of a rectangle lie on the ellipse $(x/a)^2 + (y/b)^2 = 1$. **(a)** Where should the corners be located in order to maximize the area of the rectangle? **(b)** What fraction of the area of the ellipse is covered by the rectangle with maximum area?

6-18. A particle of mass m is constrained to move under gravity with no friction on the surface $xy = z$. What is the trajectory of the particle if it starts from rest at $(x, y, z) = (1, -1, -1)$ with the z-axis vertical?