packet. As a consequence of this fact, it is the group velocity, not the phase velocity, that corresponds to the velocity at which a signal may be transmitted.*

PROBLEMS

13-1. Discuss the motion of a continuous string when the initial conditions are $\dot{q}(x, 0) = 0$, $q(x, 0) = A \sin(3\pi x/L)$. Resolve the solution into normal modes.

13-2. Rework the problem in Example 13.1 in the event that the plucked point is a distance $L/3$ from one end. Comment on the nature of the allowed modes.

13-3. Refer to Example 13.1. Show by a numerical calculation that the initial displacement of the string is well represented by the first three terms of the series in Equation 13.13. Sketch the shape of the string at intervals of time of $\frac{1}{8}$ of a period.

13-4. Discuss the motion of a string when the initial conditions are $q(x, 0) = 4x(L - x)/L^2$, $\dot{q}(x, 0) = 0$. Find the characteristic frequencies and calculate the amplitude of the nth mode.

13-5. A string with no initial displacement is set into motion by being struck over a length $2s$ about its center. This center section is given an initial velocity v_0. Describe the subsequent motion.

13-6. A string is set into motion by being struck at a point $L/4$ from one end by a triangular hammer. The initial velocity is greatest at $x = L/4$ and decreases linearly to zero at $x = 0$ and $x = L/2$. The region $L/2 \leq x \leq L$ is initially undisturbed. Determine the subsequent motion of the string. Why are the fourth, eighth, and related harmonics absent? How many decibels down from the fundamental are the second and third harmonics?

13-7. A string is pulled aside a distance h at a point $3L/7$ from one end. At a point $3L/7$ from the other end, the string is pulled aside a distance h in the opposite direction. Discuss the vibrations in terms of normal modes.

13-8. Compare, by plotting a graph, the characteristic frequencies ω_r as a function of the mode number r for a loaded string consisting of 3, 5, and 10 particles and for a continuous string with the same values of τ and $m/d = \rho$. Comment on the results.

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*The group velocity corresponds to the signal velocity only in nondispersive media (in which case the phase, group, and signal velocities are all equal) and in media of normal dispersion (in which case the phase velocity exceeds the group and signal velocities). In media with anomalous dispersion, the group velocity may exceed the signal velocity (and, in fact, may even become negative or infinite). We need only note here that a medium in which the wave number k is complex exhibits attenuation, and the dispersion is said to be *anomalous*. If k is real, there is no attenuation, and the dispersion is *normal*. What is called anomalous dispersion (due to a historical misconception) is, in fact, normal (i.e., frequent), and so-called normal dispersion is anomalous (i.e., rare). Dispersive effects are quite important in optical and electromagnetic phenomena.

Detailed analyses of the interrelationship among phase, group, and signal velocities were made by Arnold Sommerfeld and by Léon Brillouin in 1914. Translations of these papers are given in the book by Brillouin (Br60).

13-9. In Example 13.2, the complementary solution (transient part) was omitted. If transient effects are included, what are the appropriate conditions for overdamped, critically damped, and underdamped motion? Find the displacement $q(x, t)$ that results when underdamped motion is included in Example 13.2 (assume that the motion is underdamped for all normal modes).

13-10. Consider the string of Example 13.1. Show that if the string is driven at an arbitrary point, none of the normal modes with nodes at the driving point will be excited.

13-11. When a particular driving force is applied to a string, it is observed that the string vibration is purely of the nth harmonic. Find the driving force.

13-12. Determine the complementary solution for Example 13.2.

13-13. Consider the simplified wave function

$$\Psi(x, t) = Ae^{i(\omega t - kx)}$$

Assume that ω and v are complex quantities and that k is real:

$$\omega = \alpha + i\beta$$

$$v = u + iw$$

Show that the wave is damped in time. Use the fact that $k^2 = \omega^2/v^2$ to obtain expressions for α and β in terms of u and w. Find the phase velocity for this case.

13-14. Consider an electrical transmission line that has a uniform inductance per unit length L and a uniform capacitance per unit length C. Show that an alternating current I in such a line obeys the wave equation

$$\frac{\partial^2 I}{\partial x^2} - LC\frac{\partial^2 I}{\partial t^2} = 0$$

so that the wave velocity is $v = 1/\sqrt{LC}$.

13-15. Consider the superposition of two infinitely long wave trains with almost the same frequencies but with different amplitudes. Show that the phenomenon of beats occurs but that the waves never beat to zero amplitude.

13-16. Consider a wave $g(x - vt)$ propagating in the $+$ x-direction with velocity v. A rigid wall is placed at $x = x_0$. Describe the motion of the wave for $x < x_0$.

13-17. Treat the problem of wave propagation along a string loaded with particles of two different masses, m' and m'', which alternate in placement; that is,

$$m_j = \begin{cases} m', & \text{for } j \text{ even} \\ m'', & \text{for } j \text{ odd} \end{cases}$$

Show that the $\omega - k$ curve has two branches in this case, and show that there is attenuation for frequencies between the branches as well as for frequencies above the upper branch.

13-18. Sketch the phase velocity $V(k)$ and the group velocity $U(k)$ for the propagation of waves along a loaded string in the range of wave numbers $0 \leq k \leq \pi/d$. Show that $U(\pi/d) = 0$, whereas $V(\pi/d)$ does not vanish. What is the interpretation of this result in terms of the behavior of the waves?

13-19. Consider an infinitely long continuous string with linear mass density ρ_1 for $x < 0$ and for $x > L$, but density $\rho_2 > \rho_1$ for $0 < x < L$. If a wave train oscillating with an angular frequency ω is incident from the left on the high-density section of the string, find the reflected and transmitted intensities for the various portions of the string. Find a value of L that allows a maximum transmission through the high-density section. Discuss briefly the relationship of this problem to the application of nonreflective coatings to optical lenses.

13-20. Consider an infinitely long continuous string with tension τ. A mass M is attached to the string at $x = 0$. If a wave train with velocity ω/k is incident from the left, show that reflection and transmission occur at $x = 0$ and that the coefficients R and T are given by

$$R = \sin^2 \theta, \quad T = \cos^2 \theta$$

where

$$\tan \theta = \frac{M\omega^2}{2k\tau}$$

Consider carefully the boundary condition on the derivatives of the wave functions at $x = 0$. What are the phase changes for the reflected and transmitted waves?

13-21. Consider a wave packet in which the amplitude distribution is given by

$$A(k) = \begin{cases} 1, & |k - k_0| < \Delta k \\ 0, & \text{otherwise} \end{cases}$$

Show that the wave function is

$$\Psi(x, t) = \frac{2\sin[(\omega_0' t - x)\Delta k]}{\omega_0' t - x} e^{i(\omega_0 t - k_0 x)}$$

Sketch the shape of the wave packet (choose $t = 0$ for simplicity).

13-22. Consider a wave packet with a Gaussian amplitude distribution

$$A(k) = B\exp[-\sigma(k - k_0)^2]$$

where $2/\sqrt{\sigma}$ is equal to the $1/e$ width* of the packet. Using this function for $A(k)$, show that

$$\Psi(x, 0) = B\int_{-\infty}^{+\infty} \exp[-\sigma(k - k_0)^2]\exp(-ikx)\,dk$$

$$= B\sqrt{\frac{\pi}{\sigma}}\exp(-x^2/4\sigma)\exp(-ik_0 x)$$

*At the points $k = k_0 \pm 1/\sqrt{\sigma}$, the amplitude distribution is $1/e$ of its maximum value $A(k_0)$. Thus $2/\sqrt{\sigma}$ is the width of the curve at the $1/e$ height.

Sketch the shape of this wave packet. Next, expand $\omega(k)$ in a Taylor series, retain the first two terms, and integrate the wave packet equation to obtain the general result

$$\Psi(x, t) = B\sqrt{\frac{\pi}{\sigma}}\exp[-(\omega_0' t - x)^2/4\sigma]\exp[i(\omega_0 t - k_0 x)]$$

Finally, take one additional term in the Taylor series expression of $\omega(k)$ and show that σ is now replaced by a complex quantity. Find the expression for the $1/e$ width of the packet as a function of time for this case and show that the packet moves with the same group velocity as before but spreads in width as it moves. Illustrate this result with a sketch.