IF/UFRJ
Statistical Mechanics - 2025/1 – Raimundo

Problem Set #12 – Nonequilibrium Statistical Mechanics

30/6/2025

1. A trained mouse lives in the house shown in Fig. 1. A bell rings at regular intervals (much shorter than the mouse's lifetime). Each time the bell rings the mouse moves to another room; whenever it changes room, the probabilities of going through any door from the current room are the same. Determine which fraction of its lifetime will be spent on each room.

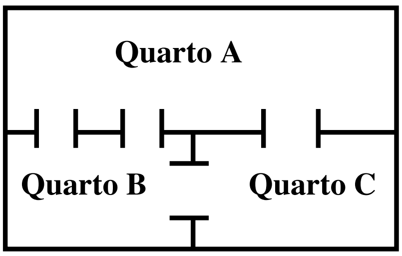

Figure 1: Problem 1 – Mouse's house.

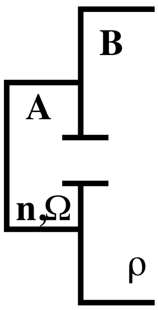

Figure 2: Prob.2 – System and reservoir.

2. Consider a box A of volume Ω connected through a small hole to another box, B, of much larger volume; see Fig. 2. Assume the probability of a particle moving from A to B in the time interval Δt is $(n/\Omega)\Delta t$, where n is the number of particles in A, and the probability of a particle moving from B to A in Δt is $\rho\Delta t$, with ρ being a constant.

   (a) Write down the master equation for the probability distribution for particles in A.

(b) Calculate the mean number of particles in A, and its variance, as functions time. Suppose that at $t = 0$, one has $n = n_0$. [*Hint: go from the master equation to the Fokker-Planck equation, and solve the latter by Fourier transform.*]

3. A spin-1/2 is in contact with a thermal bath. In the absence of an applied field, it flips between ± 1 states at a rate of $\alpha/2$ transitions per unit time, irrespective of whether it is from $+1$ to -1, or vice-versa. Let $P(\sigma, t)$ be the probability of the spin assuming the value σ at time t.

   (a) Write down a master equation for $P(\sigma, t)$, ignoring the possibility of not flipping.

   (b) Calculate the average magnetisation, $m(t) \equiv \langle \sigma \rangle$, as a function of time, assuming that σ=+1 at t=0. Sketch $m \times t$, and discuss physically your results. What is the influence of the initial condition on the behaviour at long times?

   (c) Determine $P(\sigma, t)$, subject to the initial conditions of the previous item. Sketch $P(+, t)$ and $P(-, t)$ on the same plot, and discuss physically your results. What is the influence of the initial condition on the behaviour at long times?

   Imagine now that a magnetic field is present, which tends to align the spin; the system is in contact with a thermal bath at a temperature T. The Hamiltonian describing this spin is therefore

$$\mathcal{H} = -H\sigma, \tag{1}$$

   where H is the magnetic field (in units of energy, i.e., it incorporates the magnetic moment, μ), and $\sigma = \pm 1$ denotes the spin orientation. Assume the probability per unit time that the spin is flipped, i.e. $\sigma \to -\sigma$, is given by

$$w(\sigma) = \frac{1}{2}\alpha(1 - \lambda\,\sigma), \tag{2}$$

   where $\alpha^{-1} > 0$ defines the time scale of the transitions in the absence of the field, and $\lambda > 0$ favours an imbalance between the probabilities of finding $\sigma = 1$ and $\sigma = -1$ due to the presence of the field.

   (d) Explain physically why $w(+1) < w(-1)$.

   (e) Using the fact that in equilibrium we must have

$$\frac{p(-\sigma)}{p(\sigma)} = \frac{w(\sigma)}{w(-\sigma)}, \tag{3}$$

(Why?) where $p(\sigma)$ is the Boltzmann factor, show that

$$\lambda = \tanh(H/k_{\rm B}T). \tag{4}$$

(f) Write down a master equation for $P(\sigma, t)$, ignoring the possibility of occurrence of no spin flipping.

(g) Calculate the average magnetisation, $m(t) \equiv \langle\sigma\rangle$, as a function of time, assuming that σ=+1 at t=0. Sketch $m \times t$, and discuss physically your results. What is the influence of the initial condition on the behaviour at long times?

(h) Determine $P(\sigma, t)$, subject to the initial conditions of the previous item. Sketch $P(+, t)$ and $P(-, t)$ on the same plot, and discuss physically your results. What is the influence of the initial condition on the behaviour at long times?

4. The autocorrelation function, $K(s)$, of a statistically stationary variable, $y(t)$ is given by

$$K(s) = K(0)\,\exp\left(-\alpha^2 s^2\right)\cos(2\pi f^* s). \tag{1}$$

Calculate the power spectrum, $w(f)$, and discuss its behaviour in the following limits: (i) $\alpha \to 0$; (ii) $f^* \to 0$, and (iii) both α and $f^* \to 0$.