IF/UFRJ
Statistical Mechanics - 2025/1 – Raimundo

Problem Set #11 – Phase Transitions, Part 3

23/6/2025

1. The one-dimensional Ising Hamiltonian may be written as

$$\mathcal{H} = -J \sum_i \sigma_i \sigma_{i+1}, \tag{1}$$

where $J > 0$ and $J < 0$ respectively correspond to the ferromagnetic (FM) and antiferromagnetic (AFM) systems, and $\sigma_i = \pm 1,\ \forall i$.

(a) Which are the ground states of the system in the FM and AFM cases?

(b) Define $t \equiv \tanh J/k_B T$ and $t' = \tanh(J/k_B T)'$, and show that the 'series' combination of b bonds yields $t' = t^b$; see Fig. 1. [*Hint: obtain a renormalisation group transformation (RGT) by decimating the spins shown as filled circles in Fig. 1.*]

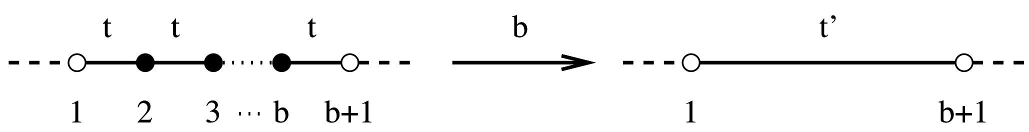

Figure 1: Problem 1: Decimation of b sites between sites 1 and $b+1$ (left panel) leads to a single bond (right panel) between them.

(c) Obtain the fixed points of the RGT. Do they depend on b?

(d) Present a detailed discussion of the physical content of each one of the fixed points, including the value of $T_{\rm c}$.

(e) Draw flow lines between the fixed points, and interpret their physical content.

(f) Comment on the adequacy (or inadequacy) to describe the antiferromagnetic case by this method.

2. Consider the cluster of sites represented in Fig. 2(a) as part of a square lattice of Ising spins; each pair of nearest neighbour spins interact via a coupling $K \equiv J/k_{\mathrm{B}}T$. Assuming 'boundary conditions' according to which sites 1 and 1' (2 and 2') are in the same spin state, the cluster is equivalent to the one in Fig. 2(b), in which external (dangling) bonds have been discarded since we are interested in probing how spin correlations spread along, say the vertical direction. A renormalisation group transformation (RGT) may be obtained through the elimination of the spin variables on sites 3 and 4 of Fig. 2(b), thus obtaining an effective coupling K' between spins on sites 1 and 2, as shown in Fig. 2(c).

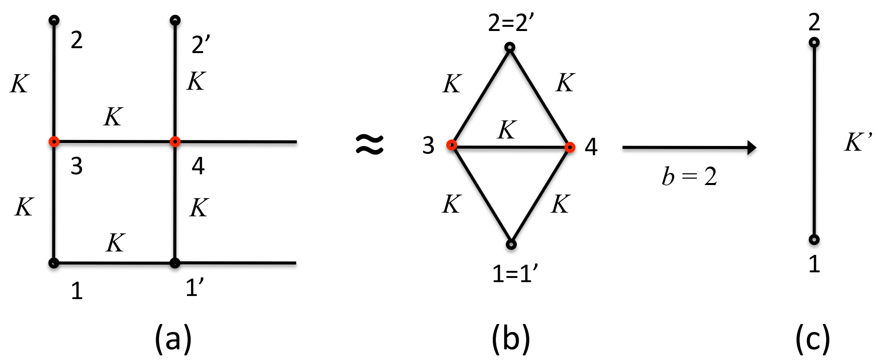

Figure 2: Problem 2 – Self-dual cluster for the square lattice.

(a) Show that the RGT in this case is given by

$$t' = \frac{2t^2(1+t)}{1+2t^3+t^4}, \tag{1}$$

where $t \equiv \tanh K$ and $t' \equiv \tanh K'$.

(b) Obtain the fixed points of the transformation and the critical point exponent ν. Compare with the exact results, $\tanh K_{\mathrm{c}} = 0.414$ and $\nu = 1$. Comment.

3. Suppose that the bonds between sites of a lattice are not all necessarily present, but they are randomly distributed, with concentration $p \in [0, 1]$.

(a) Discuss qualitatively the possibility of existence (or not) of an infinite path of sites connected by nearest neighbour bonds, in the limits $p \ll 1$ and $p \sim 1$. Establish an analogy between this *percolation* geometric transition and thermal transitions.

(b) Consider two bonds (involving three sites) arranged 'in series' as in Fig. 3(a). Obtain the probability that site 1 is connected to site 3. From your result establish the critical concentration for percolation in one dimension. Your arguments should be carefully laid out.

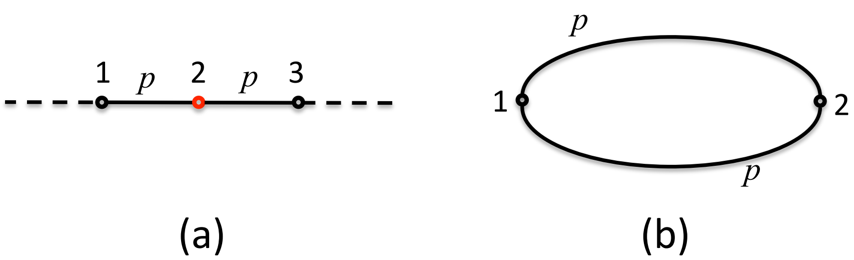

Figure 3: Problem 3, items (b) and (c) – Series (a) and parallel (b) combinations of bonds. Each bond is present with probability p.

(c) Consider the 'parallel' arrangement displayed in Fig. 3(b). Obtain the probability, p_{p}, of site 1 being connected to site 2. Your arguments should be carefully laid out.

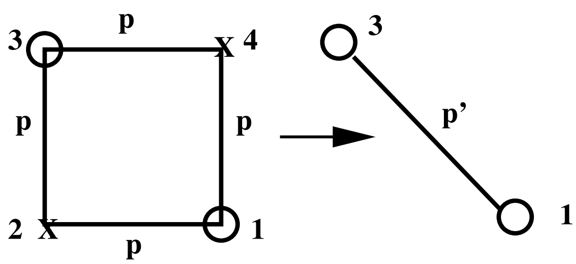

Figure 4: Problem 3, item (d) – Given that the probability of any two sites being connected by a bond is p (left panel), what is the probability, p', of 1 and 3 being connected (right panel)?

(d) Use the decimation cluster for the square lattice shown in Fig. 4 and the underlying series and parallel associations to obtain an estimate for p_{c} and ν for the bond percolation problem on a square lattice. Your arguments should be carefully laid out. Compare with the exact results, $p_{\rm c} = 1/2$ and $\nu = 4/3$.

4. *Decimation and the transfer matrix.*

   (a) We have seen that the partition function for the one-dimensional Ising model with periodic boundary conditions (PBC) may be obtained as

$$Z = \sum_{\{\sigma\}} \langle\sigma_1|T|\sigma_2\,\rangle\langle\sigma_2|T|\sigma_3\,\rangle \ldots \langle\sigma_N|T|\sigma_1\,\rangle \tag{1}$$

$$= \lambda_>^N (1 + t^N), \tag{2}$$

   where $\lambda_> = 2\cosh K$ and $\lambda_< = 2\sinh K$ are respectively the largest and smallest eigenvalues of the transfer matrix, T, $t \equiv \lambda_</\lambda_> = \tanh K$ (under no external field, as considered throughout here), and $K \equiv J/k_{\rm B}T$.

   Now consider an equivalent Ising chain, but with $N/2$ sites, with coupling K', and an added constant term to the Hamiltonian, K_0'. Show that, once the constant term is determined in terms of the unprimed variables, the condition for the partition function of these systems to be the same is

$$t' = t^2, \tag{3}$$

   where $t' \equiv \lambda'_</\lambda'_> = \tanh K'$ is the ratio of the eigenvalues of the transfer matrix for the smaller system. The procedure thus outlined therefore provides a rule to combine Ising bonds *in series* upon the elimination of an intermediate spin variable. The t-variable is usually referred to as the *transmissivity.*

   (b) Now consider the q-state Potts model on a chain with PBC,

$$\mathcal{H} = -qJ \sum_{i=1}^{N} \delta_{\sigma_i \sigma_{i+1}}, \tag{4}$$

   where $\sigma_i = 0, 1, ..., q-1$. In Problem Set #10, you have probably determined the spectrum of the transfer matrix as

$$\lambda_> = {\rm e}^{qK} + (q-1), \quad \text{non-degenerate} \tag{5}$$

$$\lambda_< = {\rm e}^{qK} - 1, \qquad (q-1)\text{-fold degenerate,} \tag{6}$$

   where $K \equiv J/k_{\rm B}T$. The ratio of these eigenvalues,

$$t \equiv \frac{\lambda_<}{\lambda_>} = \frac{1 - {\rm e}^{-qK}}{1 + (q-1){\rm e}^{-qK}}, \tag{7}$$

plays the role of the transmissivity variable for the Potts model. Show that a 'decimation' procedure similar to (a) above also yields

$$t' = t^2, \tag{8}$$

also for the Potts model. [*If you are sceptical about this procedure, you may try an explicit elimination of the middle spin, and prove that if you express the decimation transformation in terms of the* t*-variables, you recover* $t' = t^2$.]

(c) Can you provide a physical reason for the ratio of the two largest eigenvalues of the transfer matrix be so intimately connected with the scaling transformation?

5. *Duality transformation.* Let us consider a square lattice and perform a transformation consisting of three procedures, as follows.

   (i) On each *bond* of the original lattice we draw a perpendicular line, so that the intercepts of these lines also form a lattice, which we call the *dual lattice* (of the original lattice). In the case of a square lattice, Fig. 5 shows that its dual is also a square lattice, so one says that the square lattice is self dual. You may easily convince yourself that self-duality does not occur in general: for instance, the dual of the triangular lattice is the hexagonal (also called honeycomb) lattice, and vice-versa. Nonetheless, if one performs a dual transformation on the dual lattice, we recover the original lattice.

   (ii) We ascribe the same kind of spin variable to the sites of the dual lattice.

   (iii) From the thermodynamical point of view, we want to map a low-temperature phase, say in the original lattice, onto a high-temperature phase on the dual lattice, or vice-versa. That is, to the coupling $K = J/k_{\mathrm{B}}T \gg 1$ one associates a coupling $\widetilde{K} = \widetilde{J}/k_{\mathrm{B}}\widetilde{T} \ll 1$.

   Let us then consider the Ising model on a square lattice. Our previous experience suggests that the transmissivity $t \equiv \tanh K$ is more convenient to work with than the exponential function, $\exp K$.

   (a) Given that: (1) the dual variable must also be in the range $[0, 1]$; (2) if t is in the high(low)-temperature range, then $\widetilde{t}$ should be in the low(high)-temperature range; and (3) the dual of the dual is the original, show that these three properties are satisfied by the transformation

$$\widetilde{t} = \frac{1-t}{1+t}. \tag{1}$$

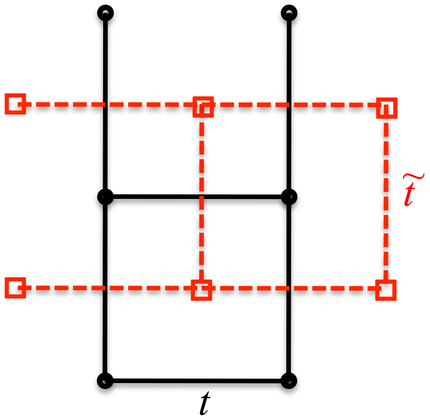

Figure 5: Problem 5 – Duality transformation: (i) to each bond in the original lattice (full black lines) we associate a perpendicular bond (dashed red lines), which therefore make up the dual lattice. (ii) to each site of the dual lattice we associate a random variable analogous to those in the original lattice.

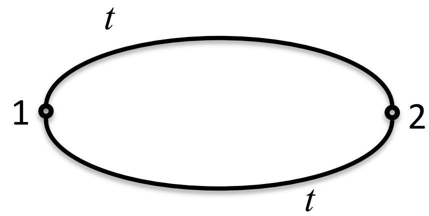

Figure 6: Problem 5 .

(b) Assuming there is a single phase transition between paramagnetic and ferromagnetic states in the whole temperature range, show that the critical point for the Ising model on the square lattice is exactly at

$$t_c = \sqrt{2} - 1. \tag{2}$$

Your arguments should be laid out carefully.

(c) Show that a parallel combination of two t's, shown in Fig. 6, yields

$$t_p = \frac{2t}{1+t^2}. \tag{3}$$

(d) Note that the dual of the cluster in Fig. 6 is a *series* combination of two $\widetilde{t}$'s. Use this fact to rederive the result (3). Your arguments should be laid out carefully.

The above results can be generalised to Potts models, with suitable modifications. With the definition of transmissivity given in Eq. (7) of Problem 4, one expects that the duality transformation must depend on q. In fact, it can be shown that the duality transformation becomes

$$\widetilde{t} = \frac{1-t}{1+(q-1)t}. \tag{4}$$

(e) Show that the three properties mentioned in item (a) are indeed satisfied by (4).

(f) Show that the exact critical temperature for the Potts model on a square lattice is

$$t_c(q) = \frac{\sqrt{q} - 1}{q - 1} \tag{5}$$

and use it to sketch $k_B T_c/J$ as a function of q.

(g) Show that a parallel combination of two Potts-model bonds is given by

$$t_p = \frac{t[2 + (q-2)t]}{1 + (q-1)t^2}. \tag{6}$$

6. *A simple decimation scheme.*

(a) Use the cluster in Fig. 7(b), together with results from Problems 4 and 5, to obtain an approximate decimation transformation for the Potts model on a square lattice.

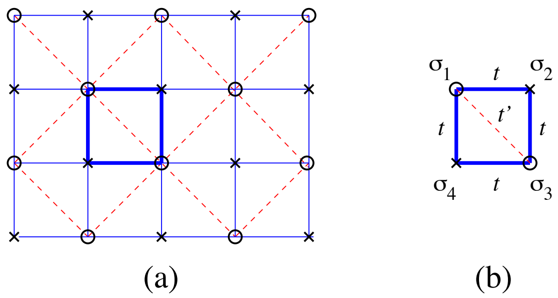

Figure 7: Problem 6 – Part of a square lattice (a) used to obtain a decimation transformation for the Potts model, through a sequence of series and parallel combinations on the cluster in (b). t is the transmissivity, and the $\sigma_i = 0, 1...(q-1)$ are the Potts variables. In (a) bonds in the original lattice are represented by full lines, decimated sites by $\times$'s, and surviving sites by circles, which also form a square lattice whose bonds are represented by dashed lines.

(b) Obtain an estimate for the fixed point, t^*, of this decimation transformation, and make a sketch of $k_B T_c/J$ as a function of q. Compare with the exact result obtained in Problem 5. Comment on your findings.

(c) Obtain an estimate for the correlation length exponent, ν, through this decimation transformation, and make a sketch of ν as a function of q. Compare with the conjecture (exact result),

$$\nu = \frac{2}{3}\left[2 + \frac{\pi}{\cos^{-1}(\frac{1}{2}\sqrt{q}) - \pi}\right]^{-1}. \tag{1}$$

Comment on your findings.