IF/UFRJ
Statistical Mechanics - 2025/1 – Raimundo

Problem Set #10 – Phase Transitions, Part 2

16/6/2025

1. (a) Using the transfer matrix method, show[1] that the correlation function for the Ising model on a one-dimensional lattice with periodic boundary conditions is given by

$$\langle\sigma_0\sigma_r\rangle = \langle\sigma_0\rangle\langle\sigma_r\rangle + a\left(\frac{\lambda_<}{\lambda_>}\right)^r, \tag{1}$$

where a is a constant, and $\lambda_>$ and $\lambda_<$ are the largest and second-largest eigenvalues of the transfer matrix; assume the magnetic field is non-zero.

(b) Then show that

$$\xi^{-1} = \ln(\lambda_>/\lambda_<). \tag{2}$$

Note that this way of calculating ξ is more general than derived here, being applicable to any 'classical' (commuting) Hamiltonian for which a transfer matrix can be defined; the ratio $\lambda_>/\lambda_<$ must then be understood as the ratio between the two largest eigenvalues in absolute value.

An example of the broader application of Eq. (2) is provided by the so-called Potts model, in which one associates a discrete classical variable, $\sigma_i = 1, 2 \ldots q$, to each site, representing the possible states of a vector. For $q = 2$, the vector can point along two directions of a straight line; it is the analogue of the z-component of a spin-1/2 operator, or the Ising model. For $q = 3, 4, 5$, etc. the vector can point to the vertices of polygons within a circle of unit radius, such as an equilateral triangle, a square, a pentagon,

[1] In your first round to solve this set, you may skip the solution to this item, since you can use Eq. (1) to show Eq. (2), and the remaining items only depend on the latter equation.

and so forth. The interaction energy between two of these vectors, located at nearest neighbour sites i and j, may be taken as

$$\mathcal{H}_{ij} = -qJ\delta_{\sigma_i\sigma_j}, \tag{3}$$

where J is a constant and $\delta_{\sigma_i\sigma_j}$ is the Kronecker δ-function. Note that for $q = 2$ this is equivalent to the Ising model, but with a coupling constant twice as large, which is immaterial for our purposes here. Consider then a one-dimensional lattice whose N ($N \gg 1$) sites are occupied by Potts variables, for the special cases $q = 3$ and 4; the magnetic field is taken as zero.

(c) Discuss the possible ground states of the system for $J > 0$. What is their degeneracy?

(d) Discuss the possible ground states of the system for $J < 0$. Is the degeneracy macroscopic for $q \geq 3$?

(e) From now on assume $J > 0$, and obtain the eigenvalues of the transfer matrix for $q = 3$ and $q = 4$.

(f) Show that the eigenvalue structure for a generic q is

$$\lambda_> = \mathrm{e}^{qJ/k_\mathrm{B}T} + (q-1), \quad \text{non-degenerate} \tag{4}$$

$$\lambda_< = \mathrm{e}^{qJ/k_\mathrm{B}T} - 1, \quad (q-1)\text{-fold degenerate.} \tag{5}$$

(g) Using Eq. (2) , obtain the low-temperature correlation length for a generic q-state Potts chain.

(h) What can you conclude about T_c for the Potts model in one dimension? Why?

(i) What can you conclude about the lower critical dimension, d_ℓ, for the Potts model? Why?

2. Consider interacting spins on a d-dimensional lattice, with linear size $L \gg 1$ (defined, in units of the lattice spacing, a, as the number of lattice sites along one of the cartesian directions). Assume one can calculate (or measure) a quantity $X_L(T)$ on this finite-size lattice; it can be the magnetisation, or the susceptibility, or the specific heat, and so forth. Also assume this quantity is such that in the thermodynamic limit one has $X \sim |T - T_c|^{-x}$, where $x = -\beta$, or γ, or α, respectively, for the examples mentioned above. The crucial observation about finite-sized systems is that near T_c (for the otherwise infinite system), there are two competing length scales, namely L and the correlation length, $\xi \sim |T - T_c|^{-\nu}$. We want to formulate a scaling theory which interpolates the

behaviour of X between the thermodynamic limit and on a finite-sized lattice. To this end, we start with the *ansatz*

$$X_L(T) = L^\omega F(L/\xi), \tag{1}$$

where ω is to be determined, and $F(y)$ is a scaling function, which must satisfy some constraints to be discussed below.

(a) If $L \gg \xi$, the correlations are not limited by the finiteness of the lattice, so that $X_L(T)$ cannot depend on L. What constraint does this impose on the behaviour of $F(y)$ for $y \gg 1$?

(b) Show that this implies in

$$\omega = x/\nu. \tag{2}$$

(c) If $L \ll \xi$, the correlations are limited by the finiteness of the lattice, so that $X_L(T)$ cannot be singular. What constraint does this impose on the behaviour of $F(y)$ for $y \ll 1$?

(d) Discuss the application of your finite-size scaling (FSS) theory to the correlation length itself. In particular, what can you predict for the ratio ξ_L/L exactly at the critical point, $T = T_c$?

3. Consider the spin-1/2 Ising model on a strip of width 2 and length N (i.e., $2 \times N$ sites), with periodic boundary conditions along the length; see Fig. 1.

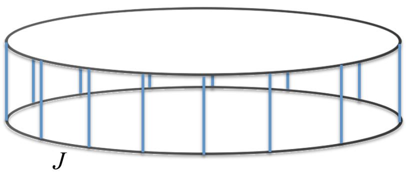

Figure 1: Problem 3 : The Ising model on a strip of width 2.

Each site is conveniently specified by the coordinates (i, j), with $i = 1, ..., N$ and $j = 1, 2$. With $\sigma_{i,j} = \pm 1$, the Hamiltonian can then be written as

$$\mathcal{H} = -J \sum_{j=1}^{2} \sum_{i=1}^{N} \sigma_{i,j}\sigma_{i+1,j} - J \sum_{i=1}^{N} \sigma_{i,1}\sigma_{i,2},$$

where the first term describes the interactions of nearest-neighbour spins on the same ring, and the second describes the interactions between nearest-neighbour spins on different rings.

(a) The transfer matrix now relates the spin configurations of a column with those of the next column. Write down the transfer matrix on a basis which makes maximum use of the symmetries of the problem. [*Hint: It is convenient to 'split' the interactions within the columns in two terms, analogously to what we did for the magnetic field term in Sec. 8.4 of the Lecture Notes; see Eq. (8.4.4).*]

(b) Obtain the correlation length at low temperatures [if necessary, resort to numerical solutions, or via Mathematica, Maple, etc.; see item (d) below].

(c) What is the critical temperature for this system?

(d) Refer to the correlation length for this strip as ξ_2, and refer to the correlation length for the linear chain as ξ_1. On the *same graph*, plot $\xi_2/2$ and ξ_1 as functions of $K \equiv J/k_{\mathrm{B}}T$.[2] Let K^* be the value of K where these two curves intersect each other. Obtain an estimate for K^*, and compare with the exact critical point for the Ising model on a *square* lattice, $K_c = 0.441$. Comment and interpret the suggested procedure.

[2] Note that one should no longer use the expressions for $\xi_{1,2}$ at low temperatures.