IF/UFRJ
Statistical Mechanics - 2025/1 – Raimundo

Problem Set #9 – Phase Transitions, Part 1

2/6/2025

1. The spin-S Heisenberg model is defined by the Hamiltonian
$$\mathcal{H} = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - \mu \sum_i \mathbf{H} \cdot \mathbf{S}_i, \tag{1}$$
where μ is the magnetic moment, $\mathbf{H}$ is an applied external field, and the sums extend over sites of a d-dimensional lattice, but $\langle ij \rangle$ restricts the sum over nearest neighbour pairs. We consider the Weiss approximation for this model.

   (a) Show that the critical temperature for magnetism is given by
   $$k_{\mathrm{B}} T_c = \frac{S(S+1)}{3} \, \frac{\mu z J}{2}, \tag{2}$$
   where z is the coordination number of the lattice (i.e. the number of nearest neighbours of any site). Compare your result with that for the Ising model (spin-S) and comment physically.

   (b) Show that the magnetisation satisfies a law of corresponding states when expressed in terms of reduced field, $\tilde{H}$, and temperature, $\tilde{T}$.

   (c) Obtain the critical exponent for the magnetisation, $M \sim (T_c - T)^{\beta}$. Does it depend on S? Does it depend on z? How does it compare with that for the Ising model, also within the Weiss approximation?

2. The simplest example of a quantum critical point (i.e., a critical point at zero temperature) is provided by systems described by the Ising model in a transverse field, Γ, whose Hamiltonian is
$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - \Gamma \sum_i \sigma_i^x \,, \tag{1}$$
where the $\boldsymbol{\sigma}$'s are Pauli matrices, and the first sum extends to pairs of nearest neighbour sites of a d-dimensional lattice with coordination number z.

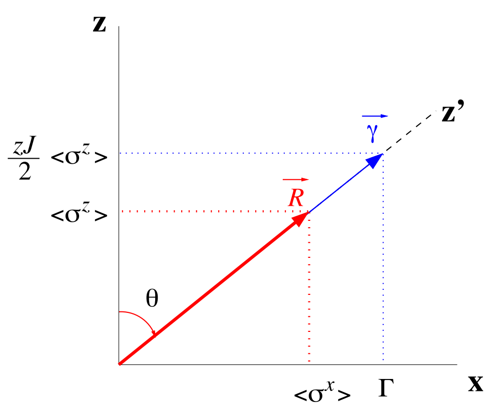

Figure 1: Problem 2 – Weiss theory for the transverse Ising model.

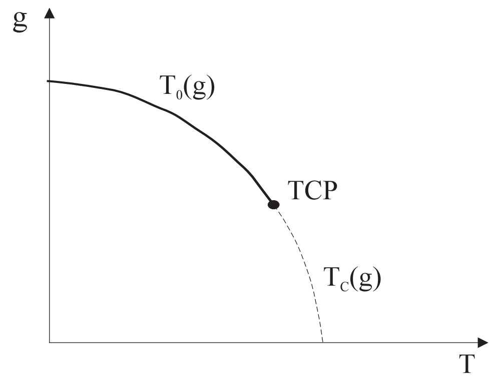

Figure 2: Problem 4 – A typical phase diagram showing a tricritical point (TCP); T is the temperature and g is a generic coupling driving the system between second- [$T_c(g)$] and first-order [$T_0(g)$] transitions.

(a) By invoking solely physical arguments,

 (i) discuss the nature of the ground state of this system in the limits $\Gamma \ll J$ and $\Gamma \gg J$;

 (ii) sketch the expected behaviour of the spontaneous magnetisation, $\langle\sigma^z\rangle$, as a function of T, for fixed Γ. Compare with what you would expect for $\langle\sigma^z\rangle(T)$ when $\Gamma = 0$.

 (iii) make a sketch of the expected behaviour of the critical temperature as a function of Γ.

In what follows we will highlight how these features can be quantitatively obtained within the Weiss approximation.

(b) We define an effective Weiss Hamiltonian as

$$\mathcal{H}_W = -\sum_i \boldsymbol{\gamma}\cdot\boldsymbol{\sigma}_i,$$

in which the mean field acting on each spin is given by

$$\boldsymbol{\gamma} = \Gamma\hat{\mathbf{x}} + \frac{zJ}{2}\langle\sigma^z\rangle\ \hat{\mathbf{z}};$$

see Fig. 1. Taking $\hat{\mathbf{z}}' \parallel \boldsymbol{\gamma}$ as the new direction of quantisation (see Fig. 1), show that

$$\langle\sigma^{z'}\rangle = \tanh\beta\gamma,$$

where $\gamma \equiv |\boldsymbol{\gamma}|$.

(c) The Weiss field, $\boldsymbol{\gamma}$, makes an angle θ with the $\hat{\mathbf{z}}$ direction, such that $\theta = 0$ when $\Gamma = 0$, or $\boldsymbol{\gamma} = (1/2)zJ\langle\sigma^z\rangle\hat{\mathbf{z}}$, and $\theta = \pi/2$ when $\langle\sigma^z\rangle = 0$, or $\boldsymbol{\gamma} = \Gamma\hat{\mathbf{x}}$; see Fig. 1. Show that γ satisfies a self-consistency condition,

$$\frac{\gamma}{\tanh\beta\gamma} = \frac{1}{2}zJ.$$

(d) The phase transition is signalled by $\langle\sigma^z\rangle \simeq 0$. Show that in this case the critical curve is given by

$$\tanh\frac{\Gamma}{k_{\mathrm{B}}T_c} = \frac{2\Gamma}{zJ},$$

and make a sketch of $\tau_c \equiv (2k_{\mathrm{B}}T_c/zJ)$ as a function of $g = 2\Gamma/zJ$. Check if your results match the qualitative prediction made in (a)(iii).

3. Consider the following expansion for the free energy in terms of the order parameter ϕ, for a magnetic system at zero field.

$$A(T,\phi) = A_0(T) + \alpha_2(T)\,\phi^2 - \alpha_4(T)\,\phi^4 + \alpha_6(T)\,\phi^6\,,$$

with $\alpha_4(T) > 0$. Suppose that near T_{c}, the coefficient of the term ϕ^6 can be written as

$$\alpha_6(T) = \frac{\alpha_4^2(T)}{3\alpha_2(T)}\,(1+\varepsilon),\ \varepsilon \equiv \frac{T - T_{\mathrm{c}}}{T_{\mathrm{c}}}\,.$$

(a) Discuss the solutions for ϕ for both $T \to T_{\mathrm{c}}^{\pm}$ (i.e. approaching T_{c} from temperatures above or below).

(b) Discuss the stability of the solutions for ϕ.

(c) Make sketches of the free energy for T above and below T_{c}. Comment on the order of the phase transition.

4. In a phase diagram, the first- and second-order critical lines smoothly meet at the so-called *tricritical point*, characterised by, say a temperature T_{t}; see Fig. 2. Within Landau's theory this point is characterised by imposing that the fourth order term in the free energy expansion vanishes identically: $\alpha_4 \equiv 0$. Let us consider a magnetic system and define $\varepsilon \equiv (T - T_{\mathrm{t}})/T_{\mathrm{t}}$. Calculate the tricritical exponents within Landau theory for the following quantities:

(a) the magnetisation, $M \sim |\varepsilon|^{\beta_t}$;

(b) the susceptibility, $\chi_T \sim |\varepsilon|^{-\gamma_t}$;

(c) the critical isotherm, $M \sim H^{1/\delta_t}$; and

(d) the specific heat, $\Delta C_H \sim |\varepsilon|^{-\alpha_t}$.

(e) *Optional. Very simple, but you'd need to read §§ 8.5 and 8.6 beforehand.* Obtain the upper critical dimension for tricritical phenomena, given that the exponents describing correlations are the same as those for the usual Landau theory, namely $\nu_{\mathrm{t}} = 1/2$ and $\eta_{\mathrm{t}} = 0$.