IF/UFRJ
Statistical Mechanics - 2025/1 – Raimundo

Problem Set #7 – Applications of Ideal Quantum Systems

19/5/2025

1. Consider a two-dimensional ideal electron gas, whose density (number of electrons by surface area of the system) is n.
   (a) Obtain the contribution from the intrinsic magnetic moments to the susceptibility of this system at $T = 0$.
   (b) Comment.

2. Show that there is no diamagnetism in Classical Physics. [Hint: The Hamiltonian for charged particles in the presence of a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is a function of $\mathbf{p}_j + (e_j/c)\mathbf{A}(\mathbf{r}_j)$. One should then show that the partition function does not depend on the applied field.]

3. Assume the conduction electrons in a metal can be treated as a Fermi gas of spin-1/2 particles, and density $n \equiv N/V$, where N and V are the total number of particles and the occupied volume, respectively; the electron mass is denoted by m. Due to the Pauli principle, the Coulomb repulsion between the electrons can be taken into account through an *effective* interaction,

$$U = \lambda \frac{N_+ N_-}{V}, \tag{1}$$

   where λ is the magnitude of the interaction, and $N_\pm$ is the number of electrons with polarisation $\sigma = \pm$; $N = N_+ + N_-$.

   (a) In the ground state one can imagine that there is a Fermi sea for each σ. Show that the Fermi vector in each case is given by

$$k_{\mathrm{F},\pm} = (6\pi^2 n_\pm)^{1/3}, \tag{2}$$

   where $n_\pm \equiv N_\pm/V$.

(b) Show that the kinetic energy density in the ground state is given by

$$\frac{E_k}{V} = \mathcal{C}\left(n_+^{5/3} + n_-^{5/3}\right), \quad \text{with} \quad \mathcal{C} \equiv \frac{3}{5}\frac{\hbar^2}{2m}(6\pi^2)^{2/3}.$$

(c) Admit that small deviations from the symmetric state can occur, *i.e.*, that

$$n_\pm = \frac{n}{2} \pm \delta, \quad 0 < \delta \ll n.$$

Discuss the physical meaning of δ, and write E_k/V up to fourth order in δ.

(d) Now express U/V in terms of n and δ.

(e) Show that the total energy density can be written in the form

$$\frac{E}{V} = e_0(n,\lambda) + w(n,\lambda)\,\delta^2 + \mathcal{O}(\delta^4).$$

and obtain expressions for the functions e_0 and w.

(f) Show that w changes sign at some $\lambda_c(n)$; determine $\lambda_c(n)$. Discuss the magnetic properties of the ground state for $w > 0$ and for $w < 0$.

(g) Sketch the behaviour of δ as a function of λ, and interpret physically. What have you learned from this very simple model?

4. Spin waves are *low-temperature* perturbations on a state with totally aligned (classical) spins [part (a) of Fig. 1]. They essentially correspond to a transverse deviation being shared by all spins; see part (c) of Fig. 1. Magnons are quanta of these excitations, which have a dispersion relation $\omega = Ak^2$, where A is a constant.

(a) Make sketches of the magnon density of modes, $\mathcal{D}(\varepsilon)$, as functions of the energy, ε, for $d = 1, 2$ and 3 spatial dimensions *in a single figure,* for comparison. Also in this figure include the average number of magnons with energy ε.

(b) Obtain the average total number of magnons for a d-dimensional system.

(c) Carefully discuss your results. In particular, comment on the consequences for the alignment when $d \leq 2$.

(d) Now admit that all density of states are displaced by $\Delta > 0$ towards the high energies, that is, $\mathcal{D}(\varepsilon) \to \mathcal{D}(\varepsilon - \Delta)$, with $\mathcal{D}(\varepsilon) \equiv 0$ for $\varepsilon < \Delta$. How would this affect the conclusions drawn in the previous item?

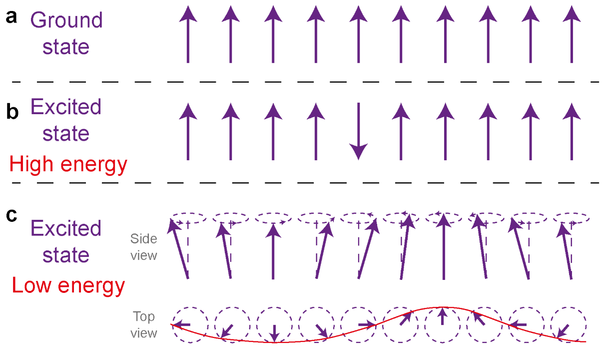

Figure 1: Problem 4. Simple picture of a Heisenberg ferromagnet. (a) Ground state: All spins are aligned. (b) A high-energy excited state, corresponding to a single spin being flipped. (c) A low-energy excited state, in which the spin flip is shared by all spins: they precess around the direction of alignment, forming a spin-wave, or magnon. Figure downloaded from https://www.rug.nl/research/zernike/physics-of-nanodevices/research/magnonspintronics

5. In some ultracold atoms experiments, N bosonic atoms of mass m move confined by a harmonic trap. In the dilute regime, the interactions between the atoms can be neglected, so that the Hamiltonian becomes

$$\mathcal{H} = \sum_{\ell=1}^{d} \sum_{i=1}^{N} \left[\frac{p_{i,\ell}^2}{2m} + \frac{1}{2} m \omega_\ell^2 x_{i,\ell}^2 \right], \tag{1}$$

written in a way that allows for d spatial dimensions, and we consider here an isotropic trap $\omega_\ell = \omega, \forall \ell = 1, ..., d$. The density of states for this system is found to be approximately given by

$$\mathcal{D}(\varepsilon) \propto \varepsilon^{d-1}. \tag{2}$$

(a) Is there Bose-Einstein condensation (BEC) for any spatial dimension, d?

(b) When applicable, determine the dependence of the critical temperature for BEC with the number of atoms.

(c) Determine the temperature dependence of the heat capacity in d dimensions at low temperatures.

(d) Compare your results in (a)-(c) with the corresponding ones in the absence of the trap.