IF/UFRJ
Statistical Mechanics - 2025/1 – Raimundo

Problem Set #6 – Quantum Effects: Bose and Fermi Statistics

28/4/2025

1. (a) Show that for ideal quantum gases, the energy density can be obtained from the grand-potential, J, as

$$\tilde{e} = -\frac{1}{V}\left(\frac{\partial}{\partial \beta}\,\beta J\right)_{z,V}. \tag{1}$$

   (b) Now suppose the single-particle levels depend on σ, $\varepsilon_{\mathbf{p}\sigma}$, to show that

$$\langle n_{\mathbf{p}'\sigma'}\rangle = -\frac{1}{\beta}\left(\frac{\partial}{\partial \varepsilon_{\mathbf{p}'\sigma'}}\,\beta J\right)_{z,T}. \tag{2}$$

2. Consider an ideal system of N particles; the states accessible to each particle have energies given by $\varepsilon_{\mathbf{p}}$ ($\mathbf{p} \equiv$ linear momentum index). Admit that each state can accommodate *up to q* particles, so that $q = 1$ corresponds to fermions, and $q = \infty$ corresponds to bosons.

   (a) Calculate the average occupation $\langle n_{\mathbf{p}}\rangle$ of the state with energy $\varepsilon_{\mathbf{p}}$; recover the limits $q \to 1$ and $q \to \infty$.

   (b) Discuss the behaviour of $\langle n_{\mathbf{p}}\rangle$ at $T = 0$.

   (c) If $1 < q < \infty$ should you expect the system to behave similarly to bosons or to fermions? Why?

3. Consider an ideal Fermi gas, with energy spectrum $\varepsilon(p) = ap^s$, contained in a hypercubic box of 'volume' $V = L^d$, in a d-dimensional space.

   (a) Show that the equation of state is

$$PV = \frac{s}{d}\,E, \tag{1}$$

   where E is the internal energy.

(b) Show that the specific heat is given by

$$\frac{C_V}{Nk_{\mathrm{B}}} = \frac{d}{s}\left(\frac{d}{s}+1\right)\frac{f_{(d/s)+1}(z)}{f_{d/s}(z)} - \left(\frac{d}{s}\right)^2 \frac{f_{d/s}(z)}{f_{(d/s)-1}(z)}, \tag{2}$$

where z is the fugacity, and

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}dx}{z^{-1}\mathrm{e}^x+1}. \tag{3}$$

(c) Obtain the low-temperature behaviour of C_V/Nk_{B}. Comment.

4. Consider an ideal gas of N bosons, with energy spectrum $\varepsilon_{\mathbf{p}} = ap^s,\ s > 0$, contained in a d-dimensional box of linear size L and volume $V = L^d$.

(a) Discuss the criterion for occurrence of Bose-Einstein condensation (BEC). Is there BEC for any value of s and d?

(b) Where applicable, determine the dependence of T_{c} with the density, $n \equiv N/V$.

(c) Discuss the dependence of the specific heat with T at low temperatures, for general s and d .