IF/UFRJ
Statistical Mechanics - 2024/1 – Raimundo

Problem Set #4 – The Canonical Ensemble – Part 2

1/4/2024

1. For a generic thermodynamical system, discuss the behaviour of E, S, C_P, C_V, α_P, $(\partial P/\partial T)_V$, and K_T as $T \to 0$.

2. Consider an ideal Maxwell-Boltzmann gas with N ($\gg 1$) particles, whose energy spectrum is $\varepsilon_{\mathbf{p}} = ap^s$, with $s > 0$ and a constants. The particles are confined to a hypecubic box of volume $V \equiv L^d$, where L is its linear size, and d is the spatial dimensionality of the box. The gas is at equilibrium at an absolute temperature T.

   (a) Obtain the canonical partition function and express it in the form

   $$Z_N(T, V, N) = \frac{1}{N!}\left(\frac{V}{\Lambda^d}\right)^N, \tag{1}$$

   with the generalised thermal wavelength, $\Lambda(T)$, being given by

   $$\Lambda \equiv \left(\frac{\beta}{\mathscr{C}}\right)^{1/s}, \tag{2}$$

   where $\mathscr{C}$ depends on constants such as the mass of the particles, m, and geometrical parameters.

   (b) Determine the dependence of the entropy with (T, V, N). Sketch $S(T)/Nk_{\mathrm{B}}$, and comment on the limiting cases $T \to 0$ and $T \to \infty$.

   (c) (i) Determine the dependence of the internal energy, E, with (N, T, V). (ii) Use the Equipartition Theorem to obtain an expression for the classical internal energy. (iii) Sketch $E(T)/Nk_{\mathrm{B}}$, and comment on the limiting cases $T \to 0$ and $T \to \infty$.

(d) Determine the dependence of the heat capacity with (N, T, V). Sketch $C_V(T)/Nk_{\mathrm{B}}$, and comment on the limiting cases $T \to 0$ and $T \to \infty$.

(e) Obtain the chemical potential $\mu(N, T, V)$. Sketch $\mu(T)$, and comment on the limiting cases $T \to 0$ and $T \to \infty$.

(f) Obtain an expression for the pressure of this gas, and relate it with the internal energy. Comment on your results.

3. Consider N uncoupled harmonic oscillators with frequency ω in the canonical ensemble.

(a) Assuming the oscillators are classical, show that the partition function is given by

$$Z(T, N) = \left(\frac{1}{u}\right)^N, \quad u \equiv \hbar\omega/k_{\mathrm{B}}T, \tag{1}$$

and calculate the following quantities: μ, P, S, E, C_P, and C_V.

(b) For quantum oscillators, show that the partition function is given by

$$Z(T, N) = [2\sinh(u/2)]^{-N}, \tag{2}$$

and calculate the same quantities as in (a).

(c) For each of the quantities calculated in (a) and (b), make sketches of their dependence with the temperature (use appropriate dimensionless variables for the axes). Plot the classical and quantum cases on the same graph, and comment on the main features and differences found.

4. A classical magnetic moment of magnitude μ can point along any spatial direction. Suppose that N identical moments like this are fixed in position on the sites of a regular lattice, but the interaction between them can be neglected. This system is in the presence of an external magnetic field $\mathbf{H} = H\hat{\mathbf{z}}$, and is at equilibrium at a temperature T.

(a) Show that the ensemble average of the z component of the magnetic moment per site is given by Langevin's expression,

$$\langle \mu_z \rangle = \mu \left[\coth\left(\frac{\mu H}{k_{\mathrm{B}}T}\right) - \frac{k_{\mathrm{B}}T}{\mu H}\right]. \tag{1}$$

(b) Now assume the system consists of spin-1/2 particles with the same magnitude, μ, of the dipole moment, subject to the same temperature T and field $\mathbf{H}$. Under what conditions is the behaviour of $\langle \mu_z \rangle$ similar to that for the classical particles of (a)? Explain this physically, and show that the two expressions are the same, apart from a numerical factor.