IF/UFRJ
Statistical Mechanics – 2024/1 – Raimundo

Problem Set #1

11/3/2022

1. Consider a box of volume V_T with N_T particles. Assume that each of the N_T particles has equal probability of being in any point of the box. A volume V within the box will contain N particles.

   (a) Obtain, for any $N \leq N_T$, the probability distribution function,
   $$f(N) \equiv f(N; V, N_T, V_T), \tag{1}$$
   of finding N particles in the volume V. Explain why you chose this distribution.

   (b) Calculate the mean value $\overline{N}$ and the variance $\overline{(N-\overline{N})^2}$. Express your results in terms of $p \equiv V/V_T$ and N_T.

   (c) Show that, for N, $N_T \gg 1$, $f(N)$ is approximately Gaussian. Note that both the average value and the variance can be read off directly from the argument of the exponential. Sketch the distribution, highlighting the most important parameters.

   (d) Show that in the limit $V/V_T \to 0$, with N_T, $V_T \to \infty$ with $N_T/V_T =$ constant, $f(N)$ approaches a Poisson distribution,
   $$f(N) = \mathrm{e}^{-\overline{N}}\,\frac{\overline{N}^{\,N}}{N!}. \tag{2}$$
   Sketch the distribution, highlighting the most important parameters. Explain when you should use this distribution or the one in (a).

   (e) Express the standard deviation (relative fluctuation),
   $$\delta \equiv \frac{\left[\overline{(N-\overline{N})^2}\right]^{1/2}}{\overline{N}}, \tag{3}$$
   in terms of N_T, p, and $q \equiv 1-p$. Now obtain numerical estimates for the fluctuations in the following specific examples:

(i) For $N_T = 10^{23}$, consider two cases: $V/V_T = 1/2$ and $V/V_T = 10^{-6}$. Comment.

(ii) For $N_T = 10$, consider two cases: $V/V_T = 1/2$ and $V/V_T = 10^{-6}$. Comment.

In view of your results for these examples, which is your overall conclusion?

2. A book with 1,400 pages has 700 typos. Under assumptions (state them explicitly!) which one should find reasonable, determine the probability that a page has 2 typos.

3. Consider a one-dimensional classical harmonic oscillator, of mass m and angular frequency ω.

   (a) Sketch the trajectory in phase space for a point representative of this oscillator.

   (b) Consider now an ensemble of these oscillators in which the initial phase is random, but uniformly distributed in the interval $[0, 2\pi]$, and an arbitrary function, $A[x(t), p(t)]$, where $x(t)$ and $p(t)$ are the position and momentum of a member of the ensemble. Show that the *time average* (over one or more periods) of $A[x(t), p(t)]$ equals its *ensemble average*. Discuss this result at length.

4. A beam of Ag atoms, each with spin-1/2, is prepared in a way that 60% of the atoms are in the $S_z = +\hbar/2$ eigenstate of $\widehat{S}_z$, and 40% are in the eigenstate $S_x = -\hbar/2$ of $\widehat{S}_x$.

   (a) Obtain the density matrix at $t = 0$, in the basis of eigenstates of $\widehat{S}_z$.

   (b) Suppose the atoms are subjected to a magnetic field $\mathbf{B} = B_0\,\hat{y}$, so that the Hamiltonian reads, $\widehat{\mathcal{H}} = \mu\,\mathbf{S}\cdot\mathbf{B}$ (μ is the magnetic moment of an atom). Obtain the density matrix at time t, in the basis of eigenstates of $\widehat{S}_z$.

   (c) Calculate $\langle S_z \rangle$ at times $t = 0$ and t.