IF/UFRJ
Graduate Quantum Mechanics I
2023/2 – Raimundo

Problem Set #12

25/10/2023 - due by 6/11/2023 at 12:00 noon

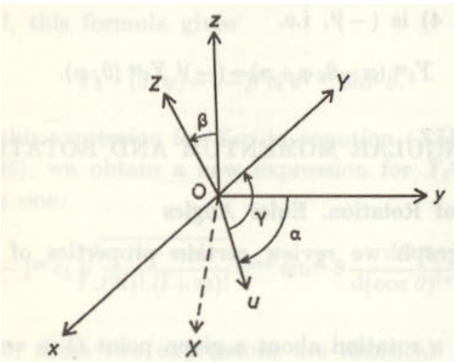

Figure 1: Problem 1: Sequence of rotations through the Euler angles; see text.

1. A rotation defined by the Euler angles, $(\alpha\beta\gamma)$, corresponds to the three successive rotations shown in Fig. 1: (1) of α around the Oz axis, such that Oy goes into Ou; (2) of β around the Ou axis, such that Oz goes into OZ; (3) of γ around the OZ axis, such that Ou goes into OY. Therefore,

$$D(\alpha\beta\gamma) = D_Z(\gamma)D_u(\beta)D_z(\alpha) = \mathrm{e}^{-(i/\hbar)\gamma J_Z}\,\mathrm{e}^{-(i/\hbar)\beta J_u}\,\mathrm{e}^{-(i/\hbar)\alpha J_z}, \tag{1}$$

where J_n is the total angular momentum component along the direction $\hat{\mathbf{n}}$. Show that $D(\alpha\beta\gamma)$ can be expressed in terms of rotations around fixed axes as

$$D(\alpha\beta\gamma) = \mathrm{e}^{-(i/\hbar)\alpha J_z}\,\mathrm{e}^{-(i/\hbar)\beta J_y}\,\mathrm{e}^{-(i/\hbar)\gamma J_z}. \tag{2}$$

2. Let $\mathbf{J}_1$ and $\mathbf{J}_2$ be two angular momentum operators, with $[J_{1\mu}, J_{2\nu}] \equiv 0$, $\forall\, \mu, \nu = x, y, z$, and $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, and let $\mathcal{R}$ be an arbitrary rotation.

(a) Show that the corresponding rotation matrices satisfy the following relation

$$\mathcal{D}^{(j)}_{m'm}(\mathcal{R}) = \sum_{m_1' m_1} \sum_{m_2' m_2} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m' \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle \times \mathcal{D}^{(j_1)}_{m_1' m_1}(\mathcal{R}) \mathcal{D}^{(j_2)}_{m_2' m_2}(\mathcal{R}),$$

where the $\langle ...|...\rangle$ are Clebsch-Gordan coefficients.

(b) Discuss this result.

3. Consider two spins-1/2, $\mathbf{S}_1$ and $\mathbf{S}_2$, fixed in position, whose interaction is described by a Hamiltonian $\mathcal{H}$. Suppose the observables $\mathbf{S}_1$ and $\mathbf{S}_2$ undergo simultaneous and identical rotations by an angle θ around a direction $\hat{\mathbf{n}}$.

(a) Show that if $\mathcal{H} = -A\mathbf{S}_1 \cdot \mathbf{S}_2$, with A a constant, then $\mathcal{H}$ is invariant for any $\hat{\mathbf{n}}$ and θ.

(b) Show that if $\mathcal{H} = -A\,(S_1^x S_2^x + S_1^y S_2^y)$, with A a constant, then $\mathcal{H}$ is invariant for $\hat{\mathbf{n}} \equiv \hat{\mathbf{z}}$ and any θ.

(c) Show that if $\mathcal{H} = -A\ S_1^z S_2^z$, with A a constant, then $\mathcal{H}$ is invariant for $\hat{\mathbf{n}} \equiv \hat{\mathbf{x}}$ (or $\hat{\mathbf{y}}$) and $\theta = \pi$.

(d) Comment about the main differences and similarities amongst your three findings above; try to identify cases in which the symmetry is discrete or continuous. Does the spin magnitude have any influence in your findings?

4. Consider rotations by an angle β around the Oy axis, for a particle with total angular momentum $\mathbf{J}$.

(a) Show that

$$\sum_{m=-j}^{j} m \left| \mathcal{D}^{(j)}_{mm'}(\beta) \right|^2 = m' \cos\beta,$$

where $\mathcal{D}^{(j)}_{mm'}(\beta)$ is the matrix representation of the corresponding rotation operator.

(b) Check your results of (a) for $j = 1/2$.

(c) Discuss the result in (a).

5. Consider a tensor operator $T_q^{(k)}$ and a state $|\alpha j m\rangle$; α represents a set of quantum numbers apart from those of angular momentum j and m. Show that $T_q^{(k)}$ acting on $|\alpha j m\rangle$ increases by q the eigenvalue of J_z.

6. Show that the matrix element of the q-th spherical tensor component of a vector operator $\mathbf{V}$ may be expressed as

$$\langle \alpha' jm' | V_q | \alpha jm \rangle = \frac{\langle \alpha' jm | \mathbf{J} \cdot \mathbf{V} | \alpha jm \rangle}{\hbar^2 j(j+1)} \langle jm' | J_q | jm \rangle,$$

where, in standard notation, $\mathbf{J}$ is the total angular momentum operator, and α denotes a set of additional quantum numbers; this result is known as *the projection theorem.*

7. *Optional.*

   (a) Express xy, xz and $(x^2 - y^2)$ as the components of an irreducible spherical tensor of rank 2.

   (b) Calculate the matrix elements

   $$e\langle \alpha, j, m' | (x^2 - y^2) | \alpha, j, m = j \rangle$$

   in terms of the quadrupole moment, given by

   $$Q \equiv e\langle \alpha, j, m = j | (3z^2 - r^2) | \alpha, j, m = j \rangle$$

   (e is the electron charge), and of the appropriate Clebsch-Gordan coefficients.

8. The electric multipole operators are defined as

$$Q_{\ell m} = \int d^3r' \, Y_\ell^m(\hat{\mathbf{r}}') \rho(\mathbf{r}'),$$

where $\rho(\mathbf{r}) = \sum_i e_i \, \delta(\mathbf{r} - \mathbf{R}_i)$ is the charge density operator associated with the charge distribution e_i, $i = 1, 2, \ldots, N$, $\delta(\mathbf{r} - \mathbf{R}_i)$ is Dirac's delta function, and the $\mathbf{R}_i$ are position operators. Show that the $Q_{\ell m}$ are irreducible tensor operators of rank ℓ. [*Hint: Examine how the $Q_{\ell m}$ transform under rotations.*]