IF/UFRJ
Graduate Quantum Mechanics I
2023/2 – Raimundo

Problem Set #11

18/10/2023 - due by 30/10/2023 at 12:00 noon

1. Let $|n\rangle$ be an eigenstate of a Hamiltonian $\mathcal{H}$, corresponding to the eigenvalue E_n, and let U be a symmetry operation which leaves $\mathcal{H}$ invariant. Show that if $U|n\rangle$ represents a state distinct from $|n\rangle$, then E_n is doubly degenerate. Generalise this result for the case of an operator $U(\lambda)$ which depends continuously on a parameter λ.

2. Consider a time-dependent infinitesimal transformation,
$$U = 1 - \frac{i}{\hbar}\, G(t)\, \delta\lambda.$$
   (a) What relation must be satisfied by the generator $G(t)$ so that the physical system is invariant under this transformation?
   (b) Obtain the generator of infinitesimal translations for a particle of mass m and charge q in the presence of an electric field $\mathbf{E}$. [*Hint: Examine the classical Hamilton's equations.*]

3. Consider the scale transformation
$$\psi(x) \longrightarrow \lambda^{1/2}\, \psi(\lambda x),\ \ \lambda \in \mathbb{R} > 0. \tag{1}$$
   (a) Show that it is unitary.
   (b) Obtain the infinitesimal generator.
   (c) From now on consider, in particular, the Hamiltonian
$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}\frac{k}{x^2}. \tag{2}$$
   Show that $\mathcal{H}' = \mathcal{H}/\lambda^2$.

(d) Show that the above transformation is a symmetry of the problem, provided time also undergoes a scale change.

(e) Obtain the explicit time dependence of the generator of this symmetry.

4. A particle moves in a periodic one-dimensional potential, $V(x \pm a) = V(x)$; physically, this may represent the motion of non-interacting electrons in a crystal lattice. Let us call $|n\rangle$, $n = 0, \pm 1, \pm 2, \ldots, \pm N$, the states representing a particle located at site n, with $\langle n'|n\rangle = \delta_{n,n'}$. Let $\mathcal{H}$ be the system Hamiltonian and $U(a)$ the discrete translation operator: $U(a)|n\rangle = |n+1\rangle$. In the tight-binding approximation, one neglects the overlap of electron states separated by a distance larger than a, so that

$$\mathcal{H}|n\rangle = E_0|n\rangle - \Delta(|n+1\rangle + |n-1\rangle) \ ,$$

where E_0 is the energy of a particle located in any site, and Δ is the energy associated with the hopping between atomic orbitals centred on lattice sites.

(a) Show that the linear combination

$$|\theta\rangle = \sum_{n=-N}^{N} \mathrm{e}^{in\theta}|n\rangle$$

is an eigenvector of both $U(a)$, with eigenvalue $\mathrm{e}^{-i\theta}$, and of $\mathcal{H}$, with eigenvalue $E(\theta) = E_0 - 2\Delta\cos\theta$.

(b) Show that the wave function associated with $|\theta\rangle$ satisfies Bloch's Theorem.