IF/UFRJ
Graduate Quantum Mechanics I
2023/2 – Raimundo

Problem Set #10

9/10/2023 - due by 16/10/2023 at 12:00 noon

1. For a spin-1/2 particle we denote by $\mathbf{S}$ its spin and by $\mathbf{L}$ its orbital angular momentum. If its state vector is $|\psi\rangle$, the two functions $\psi_+(\mathbf{r})$ and $\psi_-(\mathbf{r})$ are defined by
$$\psi_\pm(\mathbf{r}) = \langle \mathbf{r}, \pm|\psi\rangle. \tag{1}$$
Let us assume that
$$\psi_+(\mathbf{r}) = R(r)\left[Y_0^0(\theta,\varphi) + \frac{1}{\sqrt{3}}\,Y_1^0(\theta,\varphi)\right], \tag{2}$$
$$\psi_-(\mathbf{r}) = \frac{1}{\sqrt{3}}R(r)\left[Y_1^1(\theta,\varphi) - \frac{1}{\sqrt{3}}\,Y_1^0(\theta,\varphi)\right], \tag{3}$$
where (r,θ,φ) are the usual spherical coordinates and $R(r)$ is a given function of r.
   (a) What condition must $R(r)$ satisfy to render $|\psi\rangle$ normalised?
   (b) Suppose S^z is measured with the particle in the state $|\psi\rangle$. Which results can be found, and with what probabilities? Repeat this question for L^z and then for S^x.
   (c) A measurement of $\mathbf{L}^2$ is carried out with the particle in state $|\psi\rangle$, and the result is zero; which is the state just after this measurement? Repeat this question assuming the result of the measurement was $2\hbar^2$.

2. We have already seen that the angular momentum operators, J_z and $J_\pm$, when expressed on the basis $\{|\alpha j m\rangle\}$, where α represents a set of quantum numbers needed to univocally specify the basis states, they are represented by a block-diagonal matrix, each block with dimension $2j+1$. Consider then the operator
$$D_{\hat{\mathbf{n}}}(\theta) \equiv \exp\left(-i\hat{\mathbf{n}}\cdot\mathbf{J}\theta/\hbar\right), \tag{1}$$

where $\hat{\mathbf{n}}$ denotes a given spatial direction. Show that the matrices representing $D_{\hat{\mathbf{n}}}(\theta)$, on the same basis, is also block-diagonal, and with the same dimensions as the representations for $\mathbf{J}$.

3. The spin-orbit (SO) interaction in the Hydrogen atom has its origin in the fact that in its own rest frame the electron feels a magnetic field due to the proton motion; hence this field is proportional to the electron orbital angular momentum, $\mathbf{L}$. The SO interaction describes the orientational potential energy of the electron spin in this magnetic field and may be written as

$$\mathcal{H}_{\rm SO} = \lambda\, \mathbf{s}\cdot\mathbf{L}, \tag{1}$$

where $\mathbf{s}$ is the electron spin operator, and for our purposes here we may consider λ as a constant. Assuming the electron is in a state ℓ of orbital angular momentum, which are the possible energy levels of $\mathcal{H}_{\rm SO}$ and their corresponding degeneracies?

4. Consider a system with two spin-1/2 particles whose orbital variables are neglected. The Hamiltonian for this system is

$$\mathcal{H} = \omega_1 S_1^z + \omega_2 S_2^z, \tag{1}$$

where S_i^z is the projection of $\mathbf{S}_i$, $i = 1, 2$, along the direction $\hat{\mathbf{z}}$, and ω_1 and ω_2 are constants.

   (a) The state of the system at the instant $t = 0$, is

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|+-\rangle + |-+\rangle], \tag{2}$$

   in standard notation. At time t, $\mathbf{S}^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2$ is measured. What are the possible outcomes, and with what probabilities?

   (b) If the initial state of the system is arbitrary, what Bohr frequencies can appear in the evolution of $\langle \mathbf{S}^2\rangle$? Same question for $S_x = S_1^x + S_2^x$.

5. Three spin-1/2 particles are fixed in position, and let $\mathbf{S}_i$, $i = 1, 2, 3$, be their corresponding spin operators.

   (a) What is the dimension of the state space for this system?

   (b) Define

$$\mathbf{S}_{13} \equiv \mathbf{S}_1 + \mathbf{S}_3. \tag{1}$$

   Which are the possible values of the associated quantum numbers S_{13} and m_{13}?

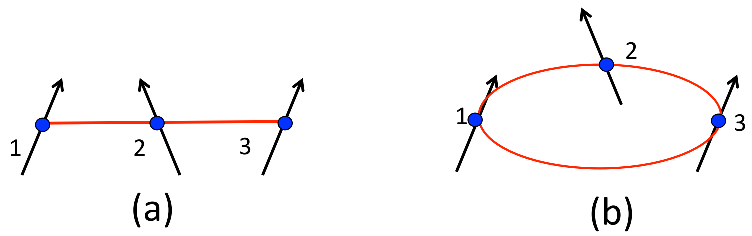

Figure 1: Problem 3 – Three interacting localised spins: (a) open ends; (b) periodic boundary conditions.

(c) The total angular momentum operator can therefore be expressed as

$$\mathbf{S} = \mathbf{S}_{13} + \mathbf{S}_2. \tag{2}$$

Which are the possible values of the associated quantum numbers S and m? How many states are generated this way?

(d) In addition to S and m, which quantum numbers must be specified in order to uniquely label the states?

(e) Assume these three spins are placed on a straight line [Fig. 1(a)], and the Hamiltonian describing their (exchange) interaction is

$$\mathcal{H}_{\text{open}} = -A\left(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3\right), \tag{3}$$

where $A < 0$ is a constant (in units of energy/$\hbar^2$). Determine the lowest energy and the ground state [expressed in terms of states labelled as described in (d)], including its degeneracy. Express the ground state(s) in the basis $|S_1 m_1 S_2 m_2 S_3 m_3\rangle$ (or $|m_1 m_2 m_3\rangle$ for short, since all $S_i = 1/2$), and discuss its (their) nature.

(f) Now assume the spins are placed on a ring [Fig. 1(b)], so that the Hamiltonian describing their (exchange) interaction is

$$\mathcal{H}_{\text{PBC}} = -A\left(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_1\right), \tag{4}$$

where $A < 0$ is a constant (in units of energy/$\hbar^2$); PBC stands for periodic boundary conditions. Determine the lowest energy and the ground state [expressed in terms of states labelled as described in (d)], including its degeneracy. Express the ground state(s) in the basis $|S_1 m_1 S_2 m_2 S_3 m_3\rangle$ (or $|m_1 m_2 m_3\rangle$ for short, since all $S_i = 1/2$), and discuss its (their) nature.