IF/UFRJ
Graduate Quantum Mechanics I
2023/2 – Raimundo

Problem Set #9

2/10/2023 - due by 16/10/2023 at 12:00 noon

1. (a) Show that if a system is in an eigenstate of J_z, then the expectation values of the operators J_x and J_y are zero.

   (b) For a system in the eigenstate $|jm\rangle$ of the operator J_z, show that the expectation value of the angular momentum along a direction z', at an angle θ with the z-axis, is $m\hbar\cos\theta$.

   (c) Show that in the state $|jm\rangle$, the largest precision in the measurement of the components J_x and J_y is obtained when $|m| = j$.

2. Consider a two-dimensional harmonic oscillator (particle of mass μ and natural frequency ω), whose Hamiltonian is

$$\mathcal{H}_{xy} = \mathcal{H}_x + \mathcal{H}_y, \tag{1}$$

with

$$\mathcal{H}_x = \frac{P_x^2}{2\mu} + \frac{1}{2}\mu\omega^2 X^2 \quad \text{and} \quad \mathcal{H}_y = \frac{P_y^2}{2\mu} + \frac{1}{2}\mu\omega^2 Y^2. \tag{2}$$

   (a) Calculate the commutators $[\mathcal{H}_x, \mathcal{H}_y]$, $[\mathcal{H}_{xy}, \mathcal{H}_x]$, $[\mathcal{H}_x, L_x]$, $[\mathcal{H}_x, L_y]$, $[\mathcal{H}_x, L_z]$, $[\mathcal{H}_{xy}, L_x]$, and $[\mathcal{H}_{xy}, L_z]$, where $\mathbf{L}$ is the orbital angular momentum operator.

   (b) Which of the following form CSCO's: $\{\mathcal{H}_{xy}\}$, $\{\mathcal{H}_x, \mathcal{H}_y\}$, $\{\mathcal{H}_{xy}, \mathcal{H}_x\}$, $\{\mathcal{H}_{xy}, L_z\}$?

   (c) Present arguments to support that the energy is given by

$$E_{xy} = (n+1)\hbar\omega, \quad \text{where } n = n_x + n_y, \text{ with } n_x, n_y = 0, 1, \ldots, \infty. \tag{3}$$

   (d) Consider the operators

$$a_x = \frac{1}{\sqrt{2}}\left(\beta X + i\frac{P_x}{\beta\hbar}\right) \quad \text{and} \quad a_y = \frac{1}{\sqrt{2}}\left(\beta Y + i\frac{P_y}{\beta\hbar}\right), \tag{4}$$

(where $\beta = \sqrt{m\omega/\hbar}$) and their hermitian conjugates $a_x^\dagger$ and $a_y^\dagger$. Calculate the commutators $[a_i, a_j^\dagger]$, with $i, j = x, y$.

(e) Now introduce the operators

$$a_r = \frac{1}{\sqrt{2}}\,(a_x - i a_y) \text{ and } a_\ell = \frac{1}{\sqrt{2}}\,(a_x + i a_y)\,, \text{ such that } N_{r,\ell} = a_{r,\ell}^\dagger a_{r,\ell}, \quad (5)$$

where the subscripts r and ℓ represent *right* and *left*, for reasons which will become clear below, and show that

(i) $[a_r, a_r^\dagger] = [a_\ell, a_\ell^\dagger] = 1$;

(ii) $\mathcal{H}_{xy} = (N_r + N_\ell + 1)\,\hbar\omega$;

(iii) $L_z = (N_r - N_\ell)\,\hbar$.

(f) What are the possible eigenvalues, n_r and n_ℓ, of N_r and N_ℓ, respectively? (*Hint: there is no need for calculations!*)

(g) An arbitrary state $|n_r n_\ell\rangle$, with n_r and n_ℓ quanta, is obtained from the vacuum, $|00\rangle$, through

$$|n_r n_\ell\rangle = \frac{1}{\sqrt{n_r! n_\ell!}}\,(a_r^\dagger)^{n_r}(a_\ell^\dagger)^{n_\ell}|00\rangle. \quad (6)$$

Show that $|n_r n_\ell\rangle$ is an eigenstate of $\mathcal{H}_{xy}$ and of L_z, with eigenvalues $(n + 1)\hbar\omega$ and $m\hbar$, respectively, with $n = n_r + n_\ell$ and $m = n_r - n_\ell$. Now explain why $a_r^\dagger$ and $a_\ell^\dagger$ create right- and left circular polarised quanta.

(h) What is the degeneracy of $\mathcal{H}_{xy}$? Show that, for a given value of n, the possible values of m are $n,\ n-2,\ n-4, \ldots, -n+2,\ -n$.

3. *Optional.* Let us now consider a three-dimensional oscillator,

$$\mathcal{H} = \mathcal{H}_x + \mathcal{H}_y + \mathcal{H}_z,$$

where $\mathcal{H}_z$ is the immediate generalisation of $\mathcal{H}_x$ (see the preceding problem) for a_z and $a_z^\dagger$.

(a) Show that $\mathcal{H}$ may be written as

$$\mathcal{H} = (N_r + N_\ell + N_z + 3/2)\,\hbar\omega.$$

(b) Show that the operators

$$L_z = \hbar\,(N_r - N_\ell)\,,$$
$$L_+ = \hbar\sqrt{2}\,\left(a_z^\dagger a_\ell - a_r^\dagger a_z\right)\,,$$
$$L_- = \hbar\sqrt{2}\left(a_\ell^\dagger a_z - a_z^\dagger a_r\right)$$

satisfy the angular momentum algebra.

(c) Show that

$$|n_r n_\ell n_z\rangle = \frac{1}{\sqrt{n_r! n_\ell! n_z!}} \left(a_r^\dagger\right)^{n_r} \left(a_\ell^\dagger\right)^{n_\ell} \left(a_z^\dagger\right)^{n_z} |000\rangle$$

are eigenstates of N_r, N_ℓ and N_z.

(d) The subspace $\mathcal{E}_n$ associated with the energy E_n is spanned by the vectors $|n_r n_\ell n_z\rangle$, such that $n_r + n_\ell + n_z = n$ and

$$L^2 |n_r n_\ell n_z\rangle = \ell(\ell+1)\hbar^2 |n_r n_\ell n_z\rangle.$$

Show that, for a given n, the possible values of ℓ are

$$\ell = n,\ n-2, \ldots,\ 0, \text{ if } n \text{ even;}$$
$$\ell = n,\ n-2, \ldots,\ 1, \text{ if } n \text{ odd.}$$

4. Consider a particle, of mass μ and charge $q < 0$, moving under the presence of a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, where the vector potential $\mathbf{A}$ is a function of the observables X, Y, Z. The Hamiltonian can be cast in a form of a free particle with the replacement $\mathbf{P} \to \mathbf{P} - q\mathbf{A}$, *i.e.*,

$$\mathcal{H} = \frac{1}{2}\mu \mathbf{V}^2,$$

where the velocity operator is given by

$$\mathbf{V} = \frac{1}{\mu}\left[\mathbf{P} - q\mathbf{A}\right].$$

Consider a uniform field $\mathbf{B} = B_0\hat{\mathbf{z}}$ and the gauge $\mathbf{A} = -\frac{1}{2}\mathbf{R} \times \mathbf{B}$.

(a) Obtain the commutation relations between the components of $\mathbf{V}$.

(b) Obtain the commutation relations $[X, V_\nu]$, with $\nu = x, y, z$; generalise for $[Y, V_\nu]$ and $[Z, V_\nu]$.

(c) The Hamiltonian may be written in the form $\mathcal{H} = \mathcal{H}_\perp + \mathcal{H}_\parallel$, with $\left[\mathcal{H}_\perp, \mathcal{H}_\parallel\right] = 0$, where $\perp$ and $\parallel$ are relative to the direction of $\mathbf{B}$. Determine $\mathcal{H}_\perp$ and $\mathcal{H}_\parallel$.

(d) What is the form of the eigenvalues of $\mathcal{H}_\parallel$? Do they form a continuum or discrete spectrum?

(e) Determine the operators Q and S, such that $[Q, S] = i$ and which allow us to write

$$\mathcal{H}_\perp = \lambda\left(Q^2 + S^2\right),$$

where λ is a constant with dimension of energy (express it in terms of the parameters of the problem!).

(f) Let $|\phi_\perp\rangle$ be an eigenstate of $\mathcal{H}_\perp$ corresponding to the eigenvalue $E_\perp$. Show that the kets

$$|\phi'_\perp\rangle = \frac{1}{\sqrt{2}}(Q + iS)|\phi_\perp\rangle \quad \text{and} \quad |\phi''_\perp\rangle = \frac{1}{\sqrt{2}}(Q - iS)|\phi_\perp\rangle$$

are eigenstates of $\mathcal{H}_\perp$, and express its eigenvalues in terms of $E_\perp$ and the other parameters. From this infer an expression for the eigenvalues $E_\perp$ of $\mathcal{H}_\perp$.

(g) Obtain an expression for the eigenvalues of $\mathcal{H}$. Discuss the degeneracies.

(h) Discuss physically the results obtained.