IF/UFRJ
Graduate Quantum Mechanics I
2023/2 – Raimundo

Problem Set #8

25/9/2023 - due by 3/9/2023 at 12:00 noon

1. In a three-dimensional problem, consider a particle of mass m and subjected to a potential energy given by

$$V(X, Y, Z) = \frac{m\omega^2}{2}\left[\left(1 + \frac{2\lambda}{3}\right)(X^2 + Y^2) + \left(1 - \frac{4\lambda}{3}\right)Z^2\right], \tag{1}$$

where the constants ω and λ satisfy

$$\omega \geq 0, \quad \text{and} \quad 0 \leq \lambda < \frac{3}{4}. \tag{2}$$

   (a) What are the eigenstates of the Hamiltonian and their corresponding energies?

   (b) Calculate and discuss, as functions of λ, the energy, the parity, and the degree of degeneracy of the ground state and the first two excited states; sketch the dependence of the energy of these states with λ.

2. Suppose that at time $t = 0$ a harmonic oscillator (mass m, angular frequency ω) is in a state

$$|\psi(0)\rangle = \sum_n c_n |n\rangle,$$

where the $|n\rangle$ are stationary states with energy $E_n = (n + 1/2)\hbar\omega$ and the c_n are the (complex) expansion coefficients, determined from specific conditions.

   (a) What is the probability, $\mathscr{P}$, that a measurement of the oscillator's energy performed at an arbitrary time $t > 0$ will yield a result greater than $2\hbar\omega$? What is the condition on the coefficients if one wants to have $\mathscr{P} = 0$?

(b) From now on, assume that only c_0 and c_1 are non-zero, such that c_0 is real and positive, and $c_1 = |c_1|e^{i\theta_1}$. By imposing that (i) $|\psi(0)\rangle$ is normalised, (ii) $\langle \mathcal{H} \rangle = \hbar\omega$, and (iii) $\langle X \rangle = 1/2\beta$, with $\beta = \sqrt{m\omega/\hbar}$, determine c_0 and c_1.

(c) Obtain the average position at time $t > 0$, $\langle X \rangle_t$. Compare with the result you would obtain if the average were taken in one of the eigenstates of the Hamiltonian.

(d) Obtain $\langle X^2 \rangle_t$ at time $t > 0$, and the uncertainty $\Delta X_t \equiv \sqrt{\langle X^2 \rangle_t - \langle X \rangle_t^2}$. Comment.

(e) Discuss your overall findings, *vis à vis* the coherent states.

3. Consider a one-dimensional harmonic oscillator, described by the Hamiltonian

$$\mathcal{H} = \hbar\omega(N + 1/2),\ N = a^+a,\ a = \sqrt{\frac{m\omega}{2\hbar}}\ X\ + \frac{i}{\sqrt{2m\hbar\omega}}\ P.$$

Let us define a unitary transformation through the operator

$$U(\lambda) = e^{-\lambda(a-a^+)},\ U^{-1}(\lambda) = U^\dagger(\lambda) =\ e^{\lambda(a-a^+)}.$$

(a) Show that $a' \equiv U(\lambda)\, a\, U^\dagger(\lambda) = a - \lambda$. It then follows trivially that $a'^+ = a^+ - \lambda$. [*Hint:* $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$, *if* $[A, B]$ *commutes with* A *and* B.]

(b) Show that $\mathcal{H}' \equiv U(\lambda)\, \mathcal{H}\, U^\dagger(\lambda) = \mathcal{H} - \lambda\hbar\omega(a + a^+) + \lambda^2\hbar\omega$.

(c) Show that if $|n\rangle$ is an eigenstate of $\mathcal{H}$, then $|n'\rangle \equiv U|n\rangle$ is an eigenstate of $\mathcal{H}'$ with the same eigenvalue: $\mathcal{H}'(U|n\rangle) = E_n(U|n\rangle)$.

(d) Give a physical interpretation for the relation between the states $|n'\rangle$ and $|n\rangle$.

(e) Use the above results to discuss the spectrum of a charged (charge q) harmonic oscillator in the presence of a uniform electric field $\mathcal{E}$.

4. A one-dimensional harmonic oscillator (mass m, charge q and frequency ω) is in the presence of an electric field $\mathcal{E}(t)$ parallel to Ox.

(a) Write down the particle Hamiltonian $\mathcal{H}(t)$ in terms of a and $a^\dagger$. Calculate the commutators of a and of $a^\dagger$ with $\mathcal{H}(t)$.

(b) Let $\alpha(t) = \langle \psi(t)|a|\psi(t) \rangle$, where $|\psi(t)\rangle$ is the normalised state of the particle. Show that $\alpha(t)$ satisfies the differential equation

$$\frac{d}{dt}\alpha(t) = -i\omega\alpha(t) + i\lambda(t),$$

where $\lambda(t) = q\mathcal{E}(t)/\sqrt{2m\hbar\omega}$. Integrate this differential equation. What are the expectation values of the position and momentum operators at time t? Comment.

(c) The ket $|\varphi(t)\rangle$ is defined by

$$|\varphi(t)\rangle = [a - \alpha(t)]|\psi(t)\rangle,$$

where $\alpha(t)$ was defined in (b). Show that the evolution of $|\varphi(t)\rangle$ is governed by

$$i\hbar\frac{d}{dt}|\varphi(t)\rangle = [\mathcal{H}(t) + \hbar\omega]|\varphi(t)\rangle.$$

How does the norm of $|\varphi(t)\rangle$ vary with time?

(d) Suppose that $|\psi(0)\rangle$ is an eigenvector of a with eigenvalue $\alpha(0)$, and show that $|\psi(t)\rangle$ is also an eigenvector of a, and obtain its eigenvalue. Determine the expectation value of the unperturbed Hamiltonian (i.e., without the coupling to the electric field), $\mathcal{H}_0$, at time t, as a function of $\alpha(0)$. Determine the root mean square deviations ΔX, ΔP, and $\Delta\mathcal{H}_0$; how do these vary with time?

(e) Suppose that at $t = 0$ the oscillator is in the ground state, $|\varphi(0)\rangle$. The electric field acts between instants 0 and T, vanishing afterwards. How do the average values $\langle X\rangle(t)$ e $\langle P\rangle(t)$ evolve when $t > T$? Aplication: suppose that between 0 e]and T we have $\mathcal{E} = \mathcal{E}_0 \cos\omega' t$; discuss the ressonances in terms of $\Delta\omega = \omega' - \omega$. If, at some instant $t > T$ the energy is measured, what values can be found, and with what probabilities?