IF/UFRJ
Graduate Quantum Mechanics I
2023/2 – Raimundo

Problem Set #6

11/9/2023 - due by 18/9/2023 at 12:00 noon

1. Consider a system with two spin-1/2 particles in a triplet state

$$|T\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_{(1)}|\downarrow\rangle_{(2)} + |\downarrow\rangle_{(1)}|\uparrow\rangle_{(2)}\right], \tag{1}$$

where $|\uparrow\rangle_{(i)}$ and $|\downarrow\rangle_{(i)}$ represent eigenstates of $S^z_{(i)}$ with eigenvalues $\pm\hbar/2$, respectively, and with $i = 1, 2$ denoting particles (1) and (2).

(a) Suppose that measurements of $S^z_{(1)}$ and $S^z_{(2)}$ are made simultaneously. Which are the possible outcomes? Once obtaining the result S ($=\uparrow$ or $\downarrow$) for particle (1), what is the probability of obtaining necessarily $-S$ for particle (2)?

(b) Is it possible to write the state $|T\rangle$ as a tensor product of states, i.e., $|T\rangle = |a\rangle_{(1)} \otimes |b\rangle_{(2)}$? If so, determine $|a\rangle_{(1)}$ and $|b\rangle_{(2)}$.

(c) Now suppose the system is in the state

$$|\psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_{(1)}|\uparrow\rangle_{(2)} + |\downarrow\rangle_{(1)}|\downarrow\rangle_{(2)} + |\uparrow\rangle_{(1)}|\downarrow\rangle_{(2)} + |\downarrow\rangle_{(1)}|\uparrow\rangle_{(2)}\right]. \tag{2}$$

Repeat item (b) for $|\psi\rangle$.

(d) Repeat item (a) for $|\psi\rangle$.

(e) Comment on the differences between $|T\rangle$ and $|\psi\rangle$.

2. Consider a system with two spin-1/2 particles.[1]

   (a) Assume the system is in the state $|\psi\rangle = |\uparrow\uparrow\rangle$. Is this an entangled state?

   (b) Write down the density matrix, ρ, in the $|\sigma_1\sigma_2\rangle$ basis, where $\sigma_i = \uparrow, \downarrow$, $i = 1, 2$.

   (c) Calculate $\text{Tr}\rho$ and ρ^2; comment.

   (d) Obtain $\rho_1 \equiv \text{Tr}_2\, \rho$, where $\rho \equiv |\psi\rangle\langle\psi|$.

   (e) Calculate $\text{Tr}_1\rho_1$ and ρ_1^2; comment.

   (f) Suppose that instead of the previous state, the system is described by

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]. \tag{1}$$

   Is this an entangled state?

   (g) Write the density matrix for this system, ρ, in the $|\sigma_1\sigma_2\rangle$ basis, where $\sigma_i = \uparrow, \downarrow$, $i = 1, 2$.

   (h) Calculate $\text{Tr}\rho$ and ρ^2; comment.

   (i) Obtain $\rho_1 \equiv \text{Tr}_2\, \rho$, where $\rho \equiv |\psi\rangle\langle\psi|$.

   (j) Calculate $\text{Tr}_1\rho_1$ and ρ_1^2; comment.

   (k) Suppose now that the density matrix is known to be

$$\rho = \frac{1}{3}\left[|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|\right], \tag{2}$$

   with

$$\begin{aligned}
|\psi_1\rangle &= |\uparrow\uparrow\rangle, \\
|\psi_2\rangle &= |\downarrow\downarrow\rangle, \\
|\psi_3\rangle &= \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right].
\end{aligned} \tag{3}$$

   Is the system in an entangled state?

   (l) Calculate $\text{Tr}\rho$ and ρ^2; comment.

   (m) Obtain $\rho_1 \equiv \text{Tr}_2\, \rho$.

   (n) Calculate $\text{Tr}_1\rho_1$ and ρ_1^2; comment.

   (o) Compare the results obtained in the three cases above, and discuss.

[1] Hint: whenever possible, try to express your 4×4 matrices in terms of 2×2 sub-matrices written in terms of the Pauli matrices, the identity matrix $\mathbb{1}$, and of the null matrix $\mathbb{0}$.

3. Consider three spin-1/2 particles fixed in position in the state

$$|\psi\rangle = \frac{1}{\sqrt{6}} \left[|{\downarrow\uparrow\uparrow}\rangle - 2|{\uparrow\downarrow\uparrow}\rangle + |{\uparrow\uparrow\downarrow}\rangle \right], \tag{1}$$

in the basis of eigenstates of $\sigma_1^z \sigma_2^z \sigma_3^z$, in standard notation.

(a) Is this a pure state? Why?

(b) Is this an entangled state? Why?

(c) Determine the partial trace over spins 2 and 3. Is the first spin in a pure state? Why?

(d) Determine the partial trace over spins 1 and 3. Is the second spin in a pure state? Why?

(e) What can you conclude from these results?

4. A system is in thermodynamic equilibrium at an absolute temperature T. Therefore, the probability of finding the system in the eigenstate $|k\rangle$, such that $\mathcal{H}|k\rangle = E_k|k\rangle$, is $p_k = \exp[-\beta E_k]/Z$, where $\beta \equiv 1/k_{\mathrm{B}}T$ and Z is the partition function; the entropy is then defined in the usual way, $S = -k_{\mathrm{B}} \sum_k p_k \ln p_k$.

(a) Show that $S \geq 0$.

(b) Show that $S = 0$ for a pure system.

(c) Supposing the Hilbert space in question is finite, with dimension N_H, show that the entropy is maximum when $p_k = 1/N_H, \ \forall k$. Discuss.

(d) Write down, in the $\{|k\rangle\}$ basis, the density operator for the situations discussed in items (b) e (c).