IF/UFRJ
Graduate Quantum Mechanics I
2023/2 – Raimundo

Problem Set #3

21/8/2023 – Due on 28/8/2023 by 12:00 noon

1. Let $|\psi\rangle$ represent an arbitrary state for a particle in three-dimensional motion.

   (a) Use the closure relation for the basis set $\{|\mathbf{p}\rangle\}$ to show that

   $$\psi(\mathbf{r}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} d^3p\, \widetilde{\psi}(\mathbf{p})\, e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}. \tag{1}$$

   (b) Use the closure relation for the basis set $\{|\mathbf{r}\rangle\}$ to show that

   $$\widetilde{\psi}(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} d^3r\, \psi(\mathbf{r})\, e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}. \tag{2}$$

   (c) Interpret these relations in terms of the superposition principle.

2. Consider the operators $\mathbf{R}$ and $\mathbf{P}$ defined by $\langle \mathbf{r}|\mathbf{R}|\psi\rangle = \mathbf{r}\,\langle \mathbf{r}|\psi\rangle$ and $\langle \mathbf{p}|\mathbf{P}|\psi\rangle = \mathbf{p}\,\langle \mathbf{p}|\psi\rangle$.

   (a) Show that $\langle \mathbf{r}|\mathbf{P}|\psi\rangle = -i\hbar\nabla_{\mathbf{r}}\langle \mathbf{r}|\psi\rangle$ and $\langle \mathbf{p}|\mathbf{R}|\psi\rangle = i\hbar\nabla_{\mathbf{p}}\langle \mathbf{p}|\psi\rangle$.

   (b) Let $f(\mathbf{R})$ be a function of the operator $\mathbf{R}$, defined by $\langle \mathbf{r}|f(\mathbf{R})|\psi\rangle = f(\mathbf{r})\langle \mathbf{r}|\psi\rangle$. Show that if $f(\mathbf{r})$ is differentiable, then $[\mathbf{P}, f(\mathbf{R})] = -i\hbar\nabla_{\mathbf{R}} f(\mathbf{R})$.

   (c) Consider now a function $g(\mathbf{P})$, and show that $[\mathbf{R}, g(\mathbf{P})] = i\hbar\nabla_{\mathbf{P}} g(\mathbf{P})$.

   (d) Show that $\mathrm{e}^{i\mathbf{P}\cdot\mathbf{a}/\hbar}\mathbf{R}\,\mathrm{e}^{-i\mathbf{P}\cdot\mathbf{a}/\hbar} = \mathbf{R} + \mathbf{a}$, with $\mathbf{a}$ being a constant vector.

   (e) Let $|\mathbf{r}_0\rangle$ be a eigenstate of $\mathbf{R}$, corresponding to the eigenvalue $\mathbf{r}_0$. Using (d), show that $\mathrm{e}^{-i\mathbf{P}\cdot\mathbf{a}/\hbar}|\mathbf{r}_0\rangle$ is an eigenstate of $\mathbf{R}$ with eigenvalue $\mathbf{r_0} + \mathbf{a}$.

   (f) Show that the wave function associated with the state $|\psi'\rangle = \mathrm{e}^{i\mathbf{P}\cdot\mathbf{a}/\hbar}|\psi\rangle$ is the same as the wave function associated with the state $|\psi\rangle$, but displaced by the vector $\mathbf{a}$; that is, $\psi'(\mathbf{r}) = \psi(\mathbf{r} + \mathbf{a})$.

(g) Discuss the physical content of the operator

$$U(\mathbf{a}) \equiv \mathrm{e}^{-i\mathbf{P}\cdot\mathbf{a}/\hbar}. \tag{1}$$

3. Consider the operator

$$R = \mathbb{1} - i\,\varepsilon\,\sigma_x, \tag{1}$$

with ε dimensionless and infinitesimal (i.e., $\varepsilon \ll 1$), and σ_x being one of the Pauli matrices, $\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$; see Problem 4 of Set #2. Define a transformation

$$\sigma'_\mu \equiv R^\dagger \sigma_\mu R, \quad \mu = x, y, z. \tag{2}$$

(a) Obtain an expression for σ'_z to first order in ε.

(b) Obtain an expression for σ'_x, to first order in ε.

(c) Obtain an expressão para σ'_y, to first order in ε.

(d) Discuss and interpret your results.

4. Consider the Pauli matrices, $\boldsymbol{\sigma} \equiv (\sigma^x, \sigma^y, \sigma^z)$, as defined in Problem 4 of Set #2; note that the component index now appears as a superscript, so that in what follows the subscript labels particles 1 and 2. In the representation in which σ^z_i, $i = 1, 2$, is diagonal, they are given by

$$\sigma^x_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y_i = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^z_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

Consider now a state space corresponding to two spin-1/2 particles, $\mathscr{E} = \mathscr{E}_1 \otimes \mathscr{E}_2$; this space has dimension $2^2 = 4$.

(a) Define the kets in a basis in $\mathscr{E}$ as $|\sigma_1\rangle \otimes |\sigma_2\rangle$, where the $|\sigma_i\rangle$ are the eigenvectors of σ^z_i, with eigenvalues $\sigma_i = \uparrow, \downarrow$ (or ± 1, if you wish), and $i = 1, 2$. Obtain a matrix representation for this basis.

(b) Using the basis of the previous item, obtain a matrix representation for $D^z \equiv \sigma^z_1 \otimes \sigma^z_2$ and for its eigenvectors. Can you relate your results to very simple procedures of setting up a matrix representation of direct products?

(c) Using the basis in (a), obtain a matrix representation for $D^x \equiv \sigma^x_1 \otimes \sigma^x_2$. Obtain the eigenvalues and eigenvectors of D^x.

(d) Using the basis in (a), obtain a matrix representation for $D^y \equiv \sigma^y_1 \otimes \sigma^y_2$. Obtain the eigenvalues and eigenvectors of D^y.

(e) Show that $[D^x, D^y] = 0$ and that $\{D^x, D^y\} = -2D^z$. You should obtain these results both by working out matrix products and by manipulating the commutators of the Pauli matrices without resorting to a representation. Compare with the corresponding results for $\boldsymbol{\sigma}_i$.