IF/UFRJ
Graduate Quantum Mechanics I
2023/2 – Raimundo

Problem Set #2

14/8/2023 – Due on 21/8/2023 by 12:00 noon

1. The state space, $\mathcal{E}$, of a certain physical system is three-dimensional. Let $\{|u_i\rangle\}, i = 1, 2, 3$, be an orthonormal basis in this space, and $\mathcal{H}$ be the system Hamiltonian. Two operators A and B are defined as follows:

$$A|u_1\rangle = |u_1\rangle,\ \ A|u_2\rangle = 0,\ \ A|u_3\rangle = -|u_3\rangle$$

$$B|u_1\rangle = |u_3\rangle,\ \ B|u_2\rangle = |u_2\rangle,\ \ B|u_3\rangle = |u_1\rangle$$

   (a) Write down the matrices representing A, B, and A^2 in the $\{|u_i\rangle\}$ basis. Are these operators observables?

   (b) Suppose $[\mathcal{H}, A^2] = 0$. Which is the most general form of $\mathcal{H}$?

   (c) Suppose $[\mathcal{H}, A] = 0$. Which is the most general form of $\mathcal{H}$?

   (d) Write down, in the $\{|u_i\rangle\}$ basis, the matrix representing the projector onto the space corresponding to the eigenvalue $+1$ of A^2.

   (e) Do A^2 and B form a CSCO ?

2. The state space, $\mathcal{E}$, of a certain physical system is four-dimensional. Let $\mathcal{H}$ be its Hamiltonian, and Π and R two observables commuting with each other, as well as with $\mathcal{H}$. A possible basis of $\mathcal{E}$ is given by the vectors $|u_i\rangle,\ i = 1, 4$, and the action of $\mathcal{H},\ \Pi$, and R on these vectors is given by

$$\mathcal{H}\ |u_1\rangle = |u_1\rangle + \omega\left(|u_3\rangle + |u_4\rangle\right)$$

$$\mathcal{H}\ |u_2\rangle = |u_2\rangle + \omega\left(|u_3\rangle + |u_4\rangle\right)$$

$$\mathcal{H}\ |u_3\rangle = \omega\left(|u_1\rangle + |u_2\rangle\right) - |u_3\rangle$$

$$\mathcal{H}\ |u_4\rangle = \omega\left(|u_1\rangle + |u_2\rangle\right) - |u_4\rangle$$

$$\Pi|u_1\rangle = |u_2\rangle, \quad \Pi|u_2\rangle = |u_1\rangle, \quad \Pi|u_3\rangle = |u_4\rangle, \quad \Pi|u_4\rangle = |u_3\rangle$$
$$R|u_1\rangle = |u_1\rangle, \quad R|u_2\rangle = |u_2\rangle, \quad R|u_3\rangle = |u_4\rangle, \quad R|u_4\rangle = |u_3\rangle$$

(a) Obtain a matrix representation for $\mathcal{H}$, Π, and R in the $\{|u_i\rangle\}$ basis.

(b) Π and R actually represent symmetries of the problem. Take this explicitly into account in order to obtain the eigenvalues of $\mathcal{H}$.

(c) Discuss if each of the following sets forms a CSCO *for any value of* ω: (i) Π and R; (ii) $\mathcal{H}$; (iii) $\mathcal{H}$, Π, and R.

3. *Optional.* Let $\mathcal{H}$ be the Hamiltonian operator, whose eigenvectors for a particular physical system are denoted by $|\varphi_n\rangle$, supposed to form a discrete orthonormal basis. Define an operator

$$U(m,n) \equiv |\varphi_m\rangle\langle\varphi_n|. \tag{1}$$

(a) Obtain the adjoint $U^\dagger(m,n)$ of $U(m,n)$.

(b) Obtain $[\mathcal{H}, U]$.

(c) Show that

$$U(m,n)U^\dagger(p,q) = \delta_{n,q}U(m,p). \tag{2}$$

(d) Obtain $\operatorname{Tr} U(m,n)$.

(e) The matrix elements of an operator A in this basis are $A_{mn} \equiv \langle\varphi_m|A|\varphi_n\rangle$. Show that

$$A = \sum_{m,n} A_{mn}U(m,n). \tag{3}$$

(f) Show that

$$A_{pq} = \operatorname{Tr}\,[AU^\dagger(p,q)]. \tag{4}$$

4. The Pauli matrices, $\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$, play a very important role in the description of spin-1/2 particles, as we will see throughout this course. In the representation in which σ_z is diagonal, $\{|\pm\rangle\}$, they are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that these matrices are Hermitian.

(b) Show that these matrices satisfy the commutation relations

$$[\sigma_k, \sigma_\ell] = 2i\varepsilon_{k\ell m}\sigma_m,$$

where the convention of summing over repeated indices is implied.

(c) Show that these matrices *anticommute*, i.e.

$$\{\sigma_k, \sigma_\ell\} \equiv \sigma_k\sigma_\ell + \sigma_\ell\sigma_k = 0,\ \ k \neq \ell.$$

(d) Show that $\sigma_k^2 = \mathbb{1},\ \ k = x, y, z$, where $\mathbb{1}$ is the 2×2 identity matrix.

(e) Verify that $\mathrm{Tr}\,\sigma_k = 0,\ \ k = x, y, z$.

(f) Define $\sigma^\pm \equiv \frac{1}{2}[\sigma_x \pm i\sigma_y]$, and show that

$$[\sigma_z, \sigma_\pm] = \pm\sigma_\pm$$

and

$$[\sigma_+, \sigma_-] = \sigma_z.$$

(g) Determine the outcomes of: (i) $\sigma^x|\pm\rangle$, (ii) $\sigma^y|\pm\rangle$, and (iii) $\sigma^\pm|\pm\rangle$.

(h) Determine the eigenvalues and eigenvectors of σ_x. Comment on the influence the fact that $\sigma_x^2 = 1$ has on the eigenvalues you found.

(i) Obtain the matrices representing the projectors onto the eigenvectors of σ_x. Check that the relations of orthogonality and completeness are satisfied.

(j) Express σ_y and σ_z on the basis of eigenvectors of σ_x. You should convince yourself that the results (a)–(f) do not depend on the basis used to represent the Pauli matrices.

(k) Show that

$$\mathrm{e}^{i\alpha\sigma_k} = \cos\alpha + i\,\sigma_k \sin\alpha,\ \ k = x, y, z,$$

where α is expressed in radians.

(l) Show that the previous result can be generalised to

$$\mathrm{e}^{i\alpha\,\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} = \cos\alpha + i\,\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\sin\alpha,$$

where $\hat{\mathbf{n}}$ is a unit vector pointing in an arbitrary direction.

5. The state space of a certain physical system is three-dimensional, and let $\{|u_i\rangle\}$, $i = 1, 3$, be an orthonormal basis in this space. Define the kets

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle, \tag{1}$$

and

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle. \tag{2}$$

(a) Are these kets normalised?

(b) Determine the matrices ρ_0 and ρ_1 which represent the projection operators onto $|\psi_0\rangle$ and $|\psi_1\rangle$, respectively. Are these matrices hermitian?

6. The state space, $\mathcal{E}$, of a certain physical system is three-dimensional. The system is described by the Hamiltonian $\mathcal{H}$, whose action on the states of an orthonormal basis in $\mathcal{E}$, $|j\rangle, j =$ 1,2, 3, is given by

$$\mathcal{H}\,|j\rangle = -\varepsilon|j\rangle - \gamma\left(|j+1\rangle + |j-1\rangle\right), \text{ with } |0\rangle \equiv |3\rangle \text{ and } |4\rangle \equiv |1\rangle;$$

ε and γ are constants. Consider an operator $\mathcal{T}$, whose action on these basis states is given by

$$\mathcal{T}|1\rangle = |2\rangle,\ \mathcal{T}|2\rangle = |3\rangle,\ \mathcal{T}|3\rangle = |1\rangle.$$

(a) Is $\mathcal{T}$ hermitian? If it is not, is it unitary? Is $\mathcal{T}$ an observable?

(b) Show that $\mathcal{T}^3 = \mathbb{1}$. From this, show that the eigenvalues of $\mathcal{T}$ are

$$\lambda_k = \omega^k,\ \text{with } k = 0, 1, 2,\ \text{and } \omega = \mathrm{e}^{i2\pi/3}.$$

What can you say about $\sum_{k=0}^{2} \lambda_k$?

(c) Show that the eigenvectors of $\mathcal{T}$ can be written as

$$|\lambda_k\rangle = \frac{1}{\sqrt{3}} \sum_{j=1}^{3} \lambda_k^{-j} |j\rangle.$$

Also show that these eigenstates form an orthonormal set.

(d) Explain why the eigenvectors of $\mathcal{T}$ form a basis.

(e) Show that the Hamiltonian may be expressed as

$$\mathcal{H} = -\varepsilon\mathbb{1} - \gamma\mathcal{T} - \gamma\mathcal{T}^2.$$

(f) Show that $[\mathcal{T}, \mathcal{H}] = 0$. Comment.

(g) Express $\mathcal{H}$ on the basis of eigenvectors of $\mathcal{T}$, and obtain the eigenenergies.

(h) Show that, in general, if $\mathcal{T}|\lambda_j\rangle = \lambda_j|\lambda_j\rangle$, with $\mathcal{T}$ unitary and such that $\lambda_j^n = 1$, with n a positive integer, and $[\mathcal{T}, \mathcal{H}] = 0$, then $\langle\lambda_j|\mathcal{H}|\lambda_k\rangle = 0$, if $\lambda_j \neq \lambda_k$. Comment.