IF/UFRJ
Graduate Quantum Mechanics I
2023/2 – Raimundo

Problem Set #1

27/7/2023 – Due on 14/8/2023 by 12:00 Noon

1. (a) Show that in one-dimensional problems the energy spectrum of bound states is always non-degenerate. *[Hint: Assume ψ_1 and ψ_2 are degenerate eigenfunctions of the time-independent Schrödinger equation, and show that $\psi_1'\psi_2 - \psi_1\psi_2' = const,$, where the primes denote first derivatives with respect to x.]*

   (b) Discuss examples verifying the above property.

2. Let $\psi(\mathbf{r}, t)$ be an eigenfunction of the Schrödinger equation corresponding to the energy E. We will see later that the *time reversed* $\psi(\mathbf{r}, t)$ is obtained by taking both $t \to -t$ and the complex conjugate of $\psi(\mathbf{r}, t)$.

   (a) Show that the function $\psi^*(\mathbf{r}, -t)$ satisfies the same Schrödinger equation as $\psi(\mathbf{r}, t)$, with the same energy E.

   (b) Consider a stationary solution, $\psi_E(\mathbf{r})\ \mathrm{e}^{-iEt/\hbar}$. Show that if the eigenvalue E is non-degenerate, then $\psi_E(\mathbf{r})$ is real, apart from an overall constant and arbitrary complex factor.

   (c) Discuss examples verifying the property derived in (b).

3. Obtain the wave functions and eigenenergies for a particle of mass m in a one-dimensional box, i.e.,

$$V = \begin{cases} 0 & \text{for } 0 \le x \le L \\ \infty & \text{otherwise.} \end{cases}$$

   Compare with the results for the case of periodic boundary conditions [i.e., $V = 0, \forall x$, but $\psi(x + L) = \psi(x)$], highlighting the main qualitative and quantitative differences.

4. *Optional.* Solve the time-independent Schrödinger equation in one dimension for the potential $V(x) = \lambda\, \delta(x)$. Consider the repulsive and attractive cases. For the attractive case, compare your results with the solutions for a square well in the limit $V_0 \to \infty$ and $a \to 0$, with $V_0 a$ finite (see, e.g., CT, Complement $\mathrm{H_I}$). For both attractive and repulsive cases, evaluate the transmission coefficient for positive energies.

5. *Optional.* Show that if the momentum distribution for a free packet, $\phi(p)$, is real and the origin is chosen so that initially $\langle x \rangle = 0$, then the equation

$$(\Delta x)^2 = (\Delta x)^2|_{t=0} \;+\; \frac{(\Delta p)^2 t^2}{m^2} \tag{1}$$

holds for a packet of arbitrary shape. [Hint: use the momentum representation.] One should notice the dependence of the width of the packet with the particle mass.

6. Consider a gaussian wave packet which, when $t = 0$, is given by

$$\psi(x, 0) = \frac{1}{(\sigma^2\pi)^{1/4}}\; \mathrm{e}^{-x^2/2\sigma^2}\; \mathrm{e}^{ik_0 x}, \tag{1}$$

where σ and k_0 are constants.

(a) Show that the momentum distribution is also gaussian;

(b) Show that this packet corresponds to the minimum uncertainty possible, $\Delta x\, \Delta p = \hbar/2$;

(c) Show that the current density when $t = 0$ is given by $j(x, 0) = \rho(x, 0)\, v_0$, where $\rho(x, 0) = |\psi(x, 0)|^2$ and $v_0 = \hbar k_0/m$.

(d) Evaluate the wave function for a free packet when $t > 0$, and show explicitly that it spreads with time according to

$$\Delta x(t) = \frac{\sigma}{\sqrt{2}}\sqrt{1 + \left(\frac{t}{\tau}\right)^2}, \tag{2}$$

where $\tau \equiv m\sigma^2/\hbar$ sets a time scale for the spread of the packet.

(e) Evaluate the current density for $t > 0$, and compare with the result obtained in (c); check what happens at the position of the packet maximum, $x_0 = v_0 t$.