Gabarito MQ2 - 2018-1 - P1

1) $[\Psi]_{l=0, j=1/2, m_j=-1/2} = \begin{pmatrix} 0 \\ Y_{00} \end{pmatrix} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

é auto-vetor de $J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$ porque $\vec{L}\, Y_{00} = 0$

2) a) $\Psi(r) = C e^{-r/a_0}$

normalização: $\int d^3r\, |\Psi|^2 = 1 \Rightarrow C^2 \int d^3r\, e^{-2r/a_0} = 1 \Rightarrow$

$C^2 \cdot 4\pi \int_0^\infty dr\, r^2 e^{-2r/a_0} = 1$

defino variável de integração $x = 2r/a_0 \Rightarrow r = \left(\frac{a_0}{2}\right) x$

$\Rightarrow \int_0^\infty dr\, r^2 e^{-2r/a_0} = \left(\frac{a_0}{2}\right)^3 \int_0^\infty dx\, x^2 e^{-x} = \frac{a_0^3}{4}$

Logo $C^2 = \frac{1}{\pi a_0^3} \Rightarrow C = \frac{1}{\sqrt{\pi a_0^3}}$

b) $\mathcal{E}^{(1)} = \langle \Psi | W_{Darwin} | \Psi \rangle$ onde $|\Psi\rangle$ é o estado não perturbado do item (a).

$\mathcal{E}^{(1)} = \frac{\pi}{2}\left(\frac{\hbar}{mc}\right)^2 e^2 \int d^3r\, |\Psi|^2 \delta^{(3)}(\vec{r}) = \frac{\pi}{2}\left(\frac{\hbar}{mc}\right)^2 e^2 \frac{1}{\pi a_0^3}$

$\mathcal{E}^{(1)} = \frac{1}{2}\alpha^2 \frac{e^2}{a_0}$ onde $\alpha = e^2/(\hbar c)$ é a const. de estrutura fina.

Temos $\mathcal{E}^{(1)}/\mathcal{E}^{(0)} = -\alpha^2$

② 3)

$$[\psi] = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{i}{2} Y_{1-1} + Y_{10} \\ -\frac{1}{2} Y_{10} + \frac{i}{\sqrt{2}} Y_{11} \end{pmatrix}$$

a)

- L^2: sim, auto-valor $\hbar^2 \cdot 1 \cdot 2 = 2\hbar^2$ porque $l=1$ em todos os termos do spinor.
- S^2: sim, auto-valor $\hbar^2 \cdot \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}\hbar^2$ porque $S^2 = \frac{3}{4}\hbar^2 \mathbb{1}$ é múltiplo da identidade
- L_z: não, porque há três valores distintos de m: $-1, 0$ e 1 na expressão de $[\psi]$
- S_z: não, porque ambas as componentes do spinor são não nulas
- J_z: não, porque cada componente envolve valores de m distintos.

b) Medida de S_x:

valor $+\hbar/2$:

$$\boxed{\cancel{P(+) = \sum_{lm} |\langle l m + | \psi \rangle|^2}}$$

$$p(+) = \sum_{lm} \left| \langle l m | {}_x\langle + | \psi \rangle \right|^2 = \sum_{m=-1}^{1} \left| \langle 1 m | {}_x\langle + | \cdot | \psi \rangle \right|^2$$

note que $|+\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, então temos

$$p(+) = \frac{1}{4} \Bigg[\underbrace{\frac{1}{4}}_{m=-1} + \underbrace{\left(1 - \frac{1}{2}\right)^2}_{m=0} + \underbrace{\frac{1}{2}}_{m=1} \Bigg] = \frac{1}{4} \left[1 \right] = \frac{1}{4}$$

$-\hbar/2$:

$$p(-) = \frac{1}{4}\Big[\underbrace{\frac{1}{4}}_{m=-1} + \underbrace{\left(1+\frac{1}{2}\right)^2}_{m=0} + \underbrace{\frac{1}{2}}_{m=+1}\Big] = \frac{1}{4}\left[\frac{3}{4}+\frac{9}{4}\right] = \frac{1}{4}\cdot 3$$

$$= 1 - p(+)$$

c) valor nulo p L_z:

$$|\psi\rangle \to |\psi\rangle' = \left(\sum_{\sigma=+,-} |10\sigma\rangle\langle 10\sigma| \cdot\right)\cdot|\psi\rangle$$

$$|\psi\rangle' = \frac{1}{\sqrt{2}}|10+\rangle - \frac{1}{2\sqrt{2}}|10-\rangle$$

normalizando: $|\psi\rangle = \dfrac{|\psi\rangle'}{\||\psi\rangle'\|} = \sqrt{\dfrac{8}{5}}\cdot\dfrac{1}{\sqrt{2}}\left(|10+\rangle - \frac{1}{2}|10-\rangle\right)$

$$= \frac{2}{\sqrt{5}}|10\rangle \otimes \left(|+\rangle_z - \frac{1}{2}|-\rangle_z\right)$$

Medida de S_x:

valor $+\hbar/2$:

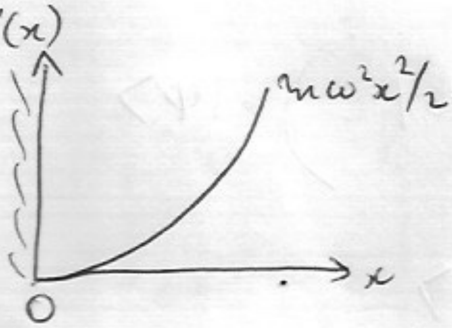
$$p(+) = \Big|\underbrace{{}_x\langle +|}_{\frac{1}{\sqrt{2}}({}_z\langle +| + {}_z\langle -|)} \cdot \frac{2}{\sqrt{5}}\left(|+\rangle_z - \frac{1}{2}|-\rangle_z\right)\Big|^2 = \frac{2}{5}\left|1-\frac{1}{2}\right|^2 = \frac{2}{5}\,\frac{1}{4} = \frac{1}{10}$$

$$p(-) = \Big|{}_x\langle -|\frac{2}{\sqrt{5}}\left(|+\rangle_z - \frac{1}{2}|-\rangle_z\right)\Big|^2 = \frac{2}{5}\left|1+\frac{1}{2}\right|^2 = \frac{2}{5}\cdot\frac{9}{4_2} = \frac{9}{10}$$

$$= 1 - p(+)$$

O emaranhamento entre os graus de liberdade orbitais e de spin presente no estado $|\psi\rangle$ implica correlação entre as medidas de L_z e S_x. As probabilidades p/ S_x dependem, assim, do valor obtido p/ L_z.

4)



$$\psi_\alpha(x) = x\, e^{-\alpha x^2}$$

a) Sol. estacionária deve satisfazer $\psi(0)=0$ devido à barreira impenetrável em $x=0$. Esta condição não é satisfeita pela família $\{\phi_\alpha(x) = e^{-\alpha x^2},\ \alpha \in \mathbb{R}^+\}$, mas é pela família $\{\psi_\alpha\}$.

b) $$\langle H\rangle = \underbrace{\left\langle \frac{p^2}{2m}\right\rangle}_{\text{cinética}} + \underbrace{\langle V\rangle}_{\text{potencial}}$$

$$\langle V\rangle = \frac{\int_0^\infty dx\, \frac{m\omega^2}{2} x^2\, x^2 e^{-2\alpha x^2}}{\underbrace{\int_0^\infty x^2 e^{-2\alpha x^2} dx}_{\frac{1}{8}\sqrt{\frac{\pi}{2\alpha^3}}}} = \frac{\frac{m\omega^2}{2}\,\frac{3}{32}\sqrt{\frac{\pi}{2\alpha^5}}}{\frac{1}{8}\sqrt{\frac{\pi}{2\alpha^3}}}$$

$$\langle V \rangle = \frac{3}{8}\frac{m\omega^2}{\alpha}$$

Cinética:

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{-\frac{\hbar^2}{2m}\int_0^\infty \Psi''(x)\,\overbrace{\Psi(x)}^{x e^{-\alpha x^2}}\,dx}{\frac{1}{8}\sqrt{\frac{\pi}{2\alpha^3}}}$$

$$\Psi' = -2\alpha x^2 e^{-\alpha x^2} + e^{-\alpha x^2} = (1-2\alpha x^2)\,e^{-\alpha x^2}$$

$$\Psi'' = -2\alpha x\,(1-2\alpha x^2)\,e^{-\alpha x^2} - 4\alpha x\,e^{-\alpha x^2}$$

$$= (4\alpha^2 x^3 - 6\alpha x)\,e^{-\alpha x^2}$$

$$\left\langle \frac{p^2}{2m} \right\rangle = -\frac{\hbar^2}{2m}\,\frac{\left(\frac{3}{8}\alpha^2\sqrt{\frac{\pi}{2\alpha^5}} - \frac{3}{4}\alpha\sqrt{\frac{\pi}{2\alpha^3}}\right)}{\frac{1}{8}\sqrt{\frac{\pi}{2\alpha^3}}}$$

$$= -\frac{\hbar^2}{2m}\left(-\frac{3}{8}\cdot 8\,\frac{\alpha^{3/2}}{\alpha^{1/2}}\right) = \frac{3}{2}\frac{\hbar^2\alpha}{m}$$

Dimensões: $(\Delta x)^2 \sim 1/\alpha \Rightarrow (\Delta p)^2 \sim \hbar^2\alpha$

Logo en. cinética $\sim \frac{\hbar^2\alpha}{m}$ cresce com α

ao passo que potencial $\sim \frac{m\omega^2}{\alpha}$ decresce.

Largura em momento $\sim \hbar\sqrt{\alpha}$, em posição $\sim 1/\sqrt{\alpha}$

e) $\langle H\rangle = \frac{3}{2}\frac{\hbar^2\alpha}{m} + \frac{3}{8}\frac{m\omega^2}{\alpha}$

$$= \frac{3}{2}\frac{\hbar^2}{m}\left(\alpha + \frac{1}{4}\frac{m^2\omega^2/\hbar^2}{\alpha}\right)$$

$$= \frac{3}{2}\frac{\hbar^2}{m}\underbrace{\left(\alpha + \frac{\alpha_0^2}{\alpha}\right)}_{f(\alpha)}, \quad \alpha_0^2 = \frac{1}{4}\left(\frac{m\omega}{\hbar}\right)^2$$

$$\alpha_0 = \frac{1}{2}\frac{m\omega}{\hbar}.$$

$\alpha_m = \alpha_0 = \frac{1}{2}\left(\frac{m\omega}{\hbar}\right)$

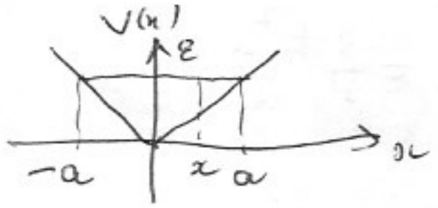
d)

$$f'(\alpha) = 1 - \frac{\alpha_0^2}{\alpha^2} = 0 \Rightarrow \alpha = \alpha_0$$

$\langle H\rangle_m = \frac{3}{2}\frac{\hbar^2}{m}2\alpha_0 = \frac{3}{2}\frac{\hbar^2}{m}\frac{m\omega}{\hbar} = \frac{3}{2}\hbar\omega$ que coincide com o resultado exato!

5) $V(x) = \alpha|x|$

a)



$\mathcal{E} = \alpha a$

$$p(x)^2 = 2m(\mathcal{E} - \alpha|x|)$$

$$p(x) = \sqrt{2m\alpha(a-|x|)}$$

b) $\int_{-a}^{a} p(x)\,dx = \sqrt{2m\alpha a}\cdot a\int_{-1}^{1} du\,\sqrt{1-|u|}$

$$= 2\sqrt{2m\alpha a^3}\underbrace{\int_0^1 du\,\sqrt{1-u}}_{2/3}$$

$$\Rightarrow \int_{-a}^{a} p(x)\, dx = \frac{4}{3}\sqrt{2m\alpha\left(\mathcal{E}/\alpha\right)^3} = \left(n+\frac{1}{2}\right)\pi\hbar$$

$$\Rightarrow \frac{4\sqrt{2}}{3}\frac{\sqrt{m}}{\alpha}\,\mathcal{E}^{3/2} = \left(n+\frac{1}{2}\right)\pi\hbar$$

$$\mathcal{E} = \left(\frac{3}{4\sqrt{2}}\frac{\alpha}{\sqrt{m}}\pi\hbar\right)^{2/3}\left(n+\frac{1}{2}\right)^{2/3}$$

c) $n \gg 1 \Rightarrow \mathcal{E} \cong \mathcal{E}_0\, n^{2/3}$

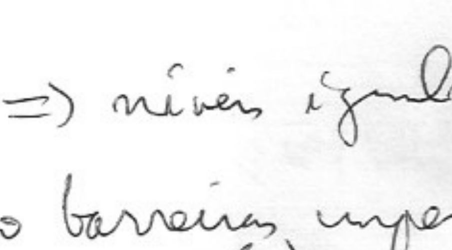
$\mathcal{E}_{n+1} - \mathcal{E}_n \simeq \frac{d\mathcal{E}}{dn} = \frac{2}{3}\mathcal{E}_0\, n^{-1/3}$ decresce com n!
espaçamento diminui c/ aument[o]

Potencial harmônico: $\mathcal{E}_n \cong \hbar\omega . n$ p/ $n \gg 1 \Rightarrow$
(b)
$\frac{d\mathcal{E}}{dn} = \hbar\omega = \mathcal{E}_{n+1} - \mathcal{E}_n$ independ[e] de n!

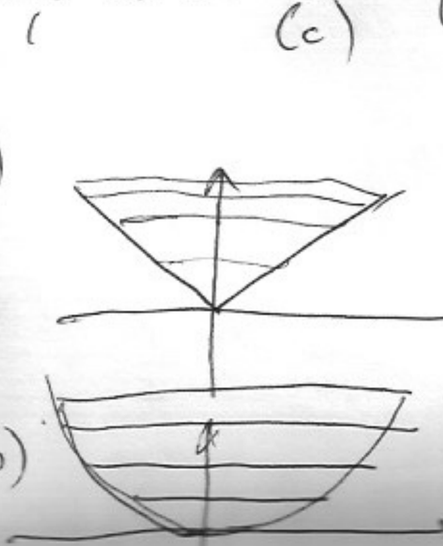
$\Rightarrow$ níveis igualmente espaçados.

Poço barreiras impenetráveis: $\mathcal{E}_n = C^{te}. n^2 \Rightarrow$
(c)
$\frac{d\mathcal{E}_n}{dn} = 2C^{te} n$ cresce c/ n.

(a)



(b)



(c)